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Explicit Time-stepping for Moving Meshes

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Abstract. In order to move the nodes in a moving mesh method a time-stepping scheme is required which is ideally explicit and non-tangling (non-overtaking in one dimension (1-D)). Such a scheme is discussed in this paper, together with its drawbacks, and illustrated in 1-D in the context of a velocity-based Lagrangian conservation method applied to first order and second order examples which exhibit a regime change after node compression. An implementation in multidimensions is also described in some detail.

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1 Adaptive moving meshes

Moving mesh methods are an alternative (or addition) to fixed mesh adaptive methods in which a given number of mesh points are relocated at each time step (also known as r-adaptivity). Relocation may be based on a velocity generated from geometric or physical principles, as in the GCL method [5] and methods based on conservation [1, 2], or on a mapping from a reference space to physical space, as in MMPDEs [6, 8, 9] and Parabolic Monge-Ampere [7] methods. Thus there is a requirement to advance the mesh in time from a given velocity or map.

In numerical implementations the size of the time step is often governed by stability considerations dependent on the numerical method used. A further challenge in advancing the mesh is the avoidance of node overtaking in 1-D or mesh tangling in 2-D. Thus time steps are sought that are not only stable but also preserve the ordering of the nodal positions in 1-D or the integrity of the mesh in higher dimensions.

For example, in one dimension, given a velocity V_j^n at a node X_j^n, $(j=0,\ldots,J)$, at time level n, the explicit Euler time stepping scheme,

$$X_j^{n+1}=X_j^n+hV_j^n, \tag{1.1}$$

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where h is the time step, is often used to update the nodes X_j^n, $(j=0,\ldots,J)$, but there is no guarantee that the ordering of the nodes will be preserved. An obvious sufficient *a priori* condition for preserving the ordering of the nodes is easily obtained from the nodal velocities and the node spacing by restricting the time step h to the shortest time that any node X_j^n takes to cross one half of either of the adjacent node spacings i.e.

$$h<\frac{1}{2}\min_j\left|\frac{\Delta X_{j\pm1/2}^m}{\Delta V_{j\pm1/2}^k}\right|$$

for all $j=0,\ldots,J$, where $\Delta X_{j\pm1/2}$ and $\Delta V_{j\pm1/2}$ denote the differences in X_j and V_j across the interval $j\pm1/2$, respectively. However, since nodes often move in concert this condition is highly restrictive and usually far from necessary. At the other exteme, a necessary time step for preserving the ordering of the nodes is obtained pragmatically by taking a speculative time step and reducing it if any node overtaking has taken place, but this is a cumbersome process and not conducive to theoretical analysis.

Implicit schemes fare better, but require more work per time step. For example, in [3] a maximum principle is used in one dimension to ensure ordering of the nodes. However, in this paper we shall only be concerned with *explicit* schemes for moving the nodes.

The layout of the paper is as follows. In the next section we introduce an explicit order-preserving scheme in 1-D and discuss its analytic basis and local truncation error. This is followed by an extension of the scheme using a higher order quadrature. In the next section two evolution problems are described to which the schemes may be applied. Numerical examples are given in Section 4 using the Lagrangian moving mesh finite difference scheme of [11, 12]. Finally, in Section 5 the extension to multidimensions is described in detail, with a summary in Section 6.

2 An explicit order-preserving scheme in 1-D

One way of achieving order-preservation of the nodes in 1-D is to focus on the *differences* $\Delta X_{j+1/2}$ between the nodal positions X_j, X_{j+1}. Applying the explicit Euler scheme (1.1) to $\Delta X_{j+1/2}$

$$\Delta X_{j+1/2}^{n+1}=\Delta X_{j+1/2}^n+h\Delta V_{j+1/2}^n=\Delta X_{j+1/2}^n\left(1+h\frac{\Delta V_{n+1/2}^n}{\Delta X_{j+1/2}^n}\right), \tag{2.1}$$

where the bracket in the final term has the status of an amplification factor. If the amplification factor becomes negative then the interval length $\Delta X_{j+1/2}$ changes sign and tangling occurs.

Suppose that the nodes are ordered at time level n so that $\Delta X_{j+1/2}^n$ is positive for all j. Then, if $\Delta V_{j+1/2}^n$ is also positive for all j, the amplification factor in (2.1) is positive and $\Delta X_{j+1/2}$ remains positive after a time step, thus preserving the ordering of the nodes.