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On the behavior of the four order iteration in Euler's family near a zero

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Abstract. The aim of this paper is to study the local convergence of the four order iteration of Euler's family for solving nonlinear operator equations. We get the optimal radius of the local convergence ball of the method for operators satisfying the weak third order generalized Lipschitz condition with $L-$average. We also show that the local convergence of the method is determined by a period 2 orbit of the method itself applied to a real function.

AMS subject classifications: 65H05, 65L20, 65N12, 39B42

Key words: local convergence; convergence ball; period 2 orbit; generalized Lipschitz condition.

1 Introduction

Let

$$f(x)=0, \tag{1.1}$$

where $f:D\subset X\to Y$ is a nonlinear operator defined on a convex set D of a real or complex Banach space X and valued in a same type space Y.

The method used often to solve a solution of (1.1) is Newton's method

$$\begin{cases} x_{n+1}=N_f^n(x), \quad x\in D,\, n\geq 1, \\ N_f(x)=x-f'(x)^{-1}f(x) \end{cases} \tag{1.2}$$

and its modifications with an approximation to f', or the modifications using higher derivatives(such as Euler's family, Halley's family and etc.).

The analysis of the convergence of an iteration is always the one of fields mostly interested in numerical nonlinear algebra. There are many papers concerning the semi-local and the local convergence of iterations for solving nonlinear equations. Parts of the latest papers are [1–6, 11].

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It is proved that for operators satisfying some kinds of Lipschitz condition with $L-$average, the criteria to guarantee the semi-local convergence of each stationary iteration of Euler's family and of Halley's family are the same, or, in other words, the criteria is an invariant independent on iterations [12–14]. The Lipschitz condition with $L-$average takes the classical Kantorovich-type condition [7] and Smale-type condition [8] as two special cases.

What interesting is that status for the local convergence is different. For Newton's method (1.2) and Euler's method

$$\begin{cases} x_{n+1}=E_f^n(x), \quad x\in D, \ n\geq 1, \\ E_f(x)=x-[1-\dfrac{1}{2}f'(x)^{-1}f''(x)f'(x)^{-1}f(x)]f'(x)^{-1}f(x). \end{cases} \tag{1.3}$$

[10] proved that the criteria to guarantee the local convergence are linked closely to the constructions of iterations and the dynamical properties of themselves applied to a real or complex function. In details, under the second order Lipschitz condition with $L-$average, [10] proved that the local convergence behavior of Newton's method is determined by a period 2 orbit‡ of itself applied to a real or complex function, and the behavior of Euler's method is determined by an repelling additional fixed point (we call $p\in X$ is an additional fixed point of $E_f(x)$, if $E_f(p)=p$ and $f(p)\neq 0$) of itself applied to the same function. If we write R_N,R_E as the optimal radii of Newton's method and Euler's method, respectively, then $R_E<R_N$.

It is natural to ask how the local convergence behavior of iterations changes along with the convergent order becomes higher. Based on that Newton's method and Euler's method are the first two members of Euler's family, in this paper, we consider the local convergence behavior of the third member of Euler's family, which is defined by

$$x_{n+1}=G_f^n(x), \quad x\in D, \ n\geq 1, \tag{1.4}$$

where

$$G_f(x)=x+\sum_{i=1}^{3}\frac{1}{i!}[f_x^{-1}(f(x))]^{(i)}(-f(x))^i,$$

and f_x^{-1} is the local inverse of f at x. [9] shows us that (1.4) can be written in the following formula:

$$\begin{aligned} G_f(x)=x+\Delta_1-\frac{1}{2}f'(x)^{-1}f''(x)\Delta_1^2-f'(x)^{-1}f''(x)\Delta_1\Delta_2 \\ -\frac{1}{3!}f'(x)^{-1}f'''(x)\Delta_1^3, \end{aligned} \tag{1.5}$$

where

$$\begin{cases} \Delta_1=\Delta_{f,1}=-f'(x)^{-1}f(x), \\ \Delta_2=\Delta_{f,2}=E_f(x)-N_f(x). \end{cases} \tag{1.6}$$

‡$\{t_1,t_2\}$, a subset of real numbers, is called a period 2 orbit of a real function $Iter(t)$ if $t_2=Iter(t_1)$ and $t_1=Iter(t_2)$. Any period 2 orbit of the iterative function is called the period 2 orbit of the corresponding iterative method.