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Highly Efficient and Accurate Spectral Approximation of the Angular Mathieu Equation for any Parameter Values q

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Abstract. The eigenpairs of the angular Mathieu equation under the periodicity condition are accurately approximated by the Jacobi polynomials in a spectral-Galerkin scheme for small and moderate values of the parameter q. On the other hand, the periodic Mathieu functions are related with the spheroidal functions of order $\pm 1/2$. It is well-known that for very large values of the bandwidth parameter, spheroidal functions can be accurately approximated by the Hermite or Laguerre functions scaled by the square root of the bandwidth parameter. This led us to employ the Laguerre polynomials in a pseudospectral manner to approximate the periodic Mathieu functions and the corresponding characteristic values for very large values of q.

AMS subject classifications: 33E10; 33D50; 65L60; 65L15

Key words: Mathieu function, spectral methods, Jacobi polynomials, Laguerre polynomials.

1 Introduction

Mathieu functions were first introduced by Mathieu in 1868 while investigating the vibrating modes of an elliptic membrane [32]. The eigenpairs of the Mathieu equation are needed in many scientific phenomena including wave motion in elliptic coordinates such as acoustic and electromagnetic scattering from an elliptic structures [14, 19, 26, 31], particle in a periodic potential [22] and vibrational spectroscopy of molecules with near resonant frequencies [37, 43]. Theoretical aspects of the Mathieu functions have been studied by many authors, including Stratton [42], McLachlan [33], Sips [41], Meixner & Schäfke [34] and Wang and Zhang [45] (cf. also [44]).

As is seen for many physical and mathematical problems in elliptic geometries the separation of variables process in elliptic coordinates leads to the Mathieu equations. If

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one wants to solve these problems with large wavenumbers, it is very important to be able to obtain accurate numerical solutions for the angular Mathieu equation for very large values of q since it is related to the wavenumber parameter present in these equations.

Mathieu functions remain difficult to employ, mainly because of the impossibility of analytically representing them in a simple and handy way [41]. There are numerous studies on the numerical computation of the Mathieu functions and corresponding eigenvalues. Erricolo used Blanch's algorithm for computing the expansion coefficients of Mathieu functions [16]. Erricolo and Carluccio provided software to compute angular and radial Mathieu functions for complex q values [17]. Shirts presented two algorithms for the computation of eigenvalues and solutions of Mathieu's differential equation for non-integer orders [39,40]. Alhargan introduced algorithms for the computation of all Mathieu functions of integer order which can deal with a range of the order n $(0-200)$ and the parameter q $(0-4n^2)$ [3]. Coïsson and co-workers describe a numerical algorithm which allows a flexible approach to the computation of all the Mathieu functions [12]. Cojocaru in [13] provided Mathieu functions computational toolbox implemented in Matlab. MATSLISE is another software package for the computation of the Mathieu eigenpairs by using the power of high-order piecewise constant perturbation methods [29] and many others [1,9,21,24,25,28,30].

Most of the above algorithms employ the well-known trigonometric series representation

$$
\begin{aligned}
ce_{2n}(\eta,q) &= \sum_{k=0}^{\infty} A_{2k}^{(2n)}(q)\cos(2k\eta), & ce_{2n+1}(\eta,q) &= \sum_{k=0}^{\infty} A_{2k+1}^{(2n+1)}(q)\cos[(2k+1)\eta], \\
se_{2n+1}(\eta,q) &= \sum_{k=0}^{\infty} B_{2k+1}^{(2n+1)}(q)\sin[(2k+1)\eta], & se_{2n+2}(\eta,q) &= \sum_{k=0}^{\infty} B_{2k+2}^{(2n+2)}(q)\sin[(2k+2)\eta], \qquad (1.1)
\end{aligned}
$$

for computing the periodic Mathieu functions where A and B are known as the expansion coefficients. There are several ways of computing these expansion coefficients such as continued fractions method [33], the forward and the backward recurrence relations [9, 17] and as the eigenvectors of tri-diagonal matrix-eigenvalue problems [12, 13]. Each has its advantages and disadvantages. However, these algorithms are not suitable for very large values of q. Thus, the aim of this study is to construct accurate and efficient spectral algorithms for the computation of the integer order periodic Mathieu functions and the corresponding characteristic values for both small and very large values of the real parameter.

The rest of the paper is organized as follows: Sections 2 and 3 are concerned with the construction of the spectral methods for small and very large values of q, respectively. Some numerical results are presented in Section 4. The last section concludes the paper with some remarks.