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IMPROVED RELAXED POSITIVE-DEFINITE AND SKEW-HERMITIAN SPLITTING PRECONDITIONERS FOR SADDLE POINT PROBLEMS*

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Abstract

We establish a class of improved relaxed positive-definite and skew-Hermitian splitting (IRPSS) preconditioners for saddle point problems. These preconditioners are easier to be implemented than the relaxed positive-definite and skew-Hermitian splitting (RPSS) preconditioner at each step for solving the saddle point problem. We study spectral properties and the minimal polynomial of the IRPSS preconditioned saddle point matrix. A theoretical optimal IRPSS preconditioner is also obtained. Numerical results show that our proposed IRPSS preconditioners are superior to the existing ones in accelerating the convergence rate of the GMRES method for solving saddle point problems.

Mathematics subject classification: 65F10, 65F50
Key words: Saddle point problems, Preconditioning, RPSS preconditioner, Eigenvalues, Krylov subspace method.

1. Introduction

We consider the iterative solution of large sparse saddle point problems of the form

$$\mathcal{A}u \equiv \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \equiv b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix, $B \in \mathbb{R}^{m\times n}$ $(m \le n)$ has full row rank, $x,\ f \in \mathbb{R}^n$ and $y,\ g \in \mathbb{R}^m$. Under these assumptions we know that the saddle point matrix $\mathcal{A}$ is nonsingular and the linear system (1.1) has a unique solution; see [1] for a general discussion about the nonsingularity of block two-by-two matrices. The saddle point problem (1.1) arises from many scientific computing and engineering applications [2], such as constrained optimization and constrained least-squares problem [3], computational fluid dynamics [4,5], data interpolation [6], element-free Galerkin discretization of elasticity problem [7–9]. The linear system (1.1) is also termed as a *Karsh-Kahn-Tucker* (KKT) system, an augmented system or an equilibrium system.

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There exists many efficient methods for solving saddle point problems, such as the null space method, the coupled direct solver, the stationary iterative method, the Krylov subspace method, and so on; see [2] for more details. The Krylov subspace method is one of the most effective methods for solving large sparse systems of linear equations [10]. However, when the Krylov subspace method is applied to solve the saddle point problem (1.1), it often converges very slowly and an efficient preconditioner is needed to achieve rapid convergence. One way to construct preconditioner is by matrix splitting iterative methods. For solving the saddle point problem (1.1), the Uzawa-like iteration methods [2] and the *Hermitian and skew-Hermitian splitting* (HSS)-like iteration methods [11–13] are two classes of efficient iterative methods, which lead to the block diagonal and block triangular preconditioners [14–16] and the HSS-like preconditioners [12, 13], respectively.

Let

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ -B & 0 \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} S & B^T \\ -B & 0 \end{bmatrix} = \mathcal{H} + \mathcal{S}$$

be the Hermitian and skew-Hermitian splitting of the matrix $\mathcal{A}$, where $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$ are the symmetric and the skew-symmetric parts of the (1,1) block matrix A, respectively. Applying the HSS iteration method proposed in [11], Benzi and Golub [13] proposed the HSS preconditioner

$$\mathcal{P}_{HSS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + H & 0 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} \alpha I + S & B^T \\ -B & \alpha I \end{bmatrix} \tag{1.2}$$

for the saddle point problem (1.1), where α is a real positive parameter and I is the identity matrix of suitable dimension. The HSS preconditioner (1.2) is induced by the stationary HSS iteration method

$$\begin{cases} (\alpha I + \mathcal{H})u^{k+\frac{1}{2}} = (\alpha I - \mathcal{S})u^k + b, \\ (\alpha I + \mathcal{S})u^{k+1} = (\alpha I - \mathcal{H})u^{k+\frac{1}{2}} + b, \end{cases} \qquad k = 0, 1, 2, \ldots.$$

It is noted that Bai et al. [11] first proposed the HSS iteration method for solving non-Hermitian positive definite linear systems and they proved the unconditionally convergent property of this method. Then Benzi and Golub [13] applied the HSS iteration method to the saddle point problem (1.1) and proved that it is also unconditionally convergent for (1.1). There are several variants of the HSS iteration method in recent years; see [17–21]. Spectral properties of the HSS preconditioned matrices as well as the optimal parameters can be found in [22–27].

The HSS iteration method is a two-half steps iteration method. The first step is easy to solve since $\alpha I + \mathcal{H}$ is symmetric positive definite. However, the second step is difficult to solve since the coefficient matrix $\alpha I + \mathcal{S}$ has the same structure as the original saddle point matrix $\mathcal{A}$ and the (1,1) block of the matrix $\alpha I + \mathcal{S}$ is also nonsymmetric. Based on the idea of the *positive-definite and skew-Hermitian splitting* (PSS) iteration method [28], Pan et al. [29] proposed a *deteriorated PSS* (DPSS) preconditioner

$$\widetilde{\mathcal{P}}_{DPSS} = \frac{1}{2\alpha} \begin{bmatrix} \alpha I + A & 0 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} \alpha I & B^T \\ -B & \alpha I \end{bmatrix}$$

for the saddle point problem (1.1). When A is Hermitian, the DPSS preconditioner is the same as the HSS preconditioner. When A is non-Hermitian, the DPSS preconditioner is easier