Journal of Computational Mathematics
Vol.37, No.1, 2019, 61–75.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1710-m2017-0175

THE HIGH ORDER BLOCK RIP CONDITION FOR SIGNAL RECOVERY*

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Abstract

In this paper, we consider the recovery of block sparse signals, whose nonzero entries appear in blocks (or clusters) rather than spread arbitrarily throughout the signal, from incomplete linear measurements. A high order sufficient condition based on block RIP is obtained to guarantee the stable recovery of all block sparse signals in the presence of noise, and robust recovery when signals are not exactly block sparse via mixed l_2/l_1 minimization. Moreover, a concrete example is established to ensure the condition is sharp. The significance of the results presented in this paper lies in the fact that recovery may be possible under more general conditions by exploiting the block structure of the sparsity pattern instead of the conventional sparsity pattern.

Mathematics subject classification: 90C59, 94A12, 94A20.
Key words: Block sparsity, Block restricted isometry property, Compressed sensing, Mixed l_2/l_1 minimization.

1. Introduction

Compressed sensing (CS), a new type of sampling theory, is a fast growing field of research. It has attracted considerable interest in a number of fields including applied mathematics, statistics, seismology, signal processing and electrical engineering. Interesting applications include radar system [26,50], coding theory [1,13], DNA microarrays [39], color imaging [33], magnetic resonance imaging [31]. Up to now, there are already many works on CS [3,4,15–17,29,30,40–44]. The key problem in CS is to recover an unknown high-dimensional sparse signal $x \in \mathbb{R}^N$ using an efficient algorithm through a sensing matrix $A \in \mathbb{R}^{n\times N}$ and the following linear measurements

$$y = Ax + z \tag{1.1}$$

where the observed signal $y \in \mathbb{R}^n$, $n \ll N$ and the vector of measurement errors $z \in \mathbb{R}^n$. In general, the solutions to the underdetermined system of linear equations (1.1) are not unique. But now suppose that x is known to be sparse in the sense that it contains only a small number of nonzero entries, which can occur in anywhere in x. This premise fundamentally changes the problem such that there is a unique sparse solution under regularity conditions. It is well known the l_1 minimization approach, a widely used algorithm, is an effective way to recover sparse signals in many settings. One of the most widely used frameworks to depict recovery

* Received July 7, 2017 / Revised version received September 14, 2017 / Accepted October 10, 2017 / Published online August 22, 2018 /

ability of l_1 minimization in CS is the restricted isometry property (RIP) introduced by Candès and Tao [13]. Let $A \in \mathbb{R}^{n\times N}$ be a matrix and $1 \le k \le N$ is an integer, the restricted isometry constant (RIC) δ_k of order k is defined as the smallest nonnegative constant that satisfies

$$(1-\delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\delta_k)\|x\|_2^2,$$

for all $k-$sparse vectors $x \in \mathbb{R}^N$. A vector $x \in \mathbb{R}^N$ is $k-$sparse if $|\mathrm{supp}(x)| \le k$, where $\mathrm{supp}(x) = \{i : x_i \neq 0\}$ is the support of x. When k is not an integer, we define δ_k as $\delta_{\lceil k\rceil}$. It has been shown l_1 minimization can recover a sparse signal with a small or zero error under some appropriate RIC met by the measurement matrix A [5–12, 23, 24, 37]. As far as we know, a sharp sufficient condition based on RIP for exact and stable recovery of signals in both noiseless and noisy cases by l_1 minimization was established by Cai and Zhang [8].

However, in practical examples, there are signals which have a particular sparsity pattern, where the nonzero coefficients appear in some blocks (or clusters). Such signals are referred to as block sparse [19, 20, 46]. In practice, the block sparse structure is very common, such as reconstruction of multi-band signals [35], equalization of sparse communication channels [18] and multiple measurement vector (MMV) model [20, 21, 36]. Actually, the notion of block sparsity was already introduced in statistics literature and was named the group Lasso estimator [2,14,27,34,38,48]. Recently, block sparsity pattern has attracted significant attention. Various efficient methods and explicit recovery guarantees [19,20,22,25,28,32,45–47] have been proposed.

In this paper, our goal is to recover the unknown signal x from linear measurements (1.1). But at the moment, nonzero elements of signal x are occurring in blocks (or clusters) instead of spreading arbitrarily throughout the signal vector. To this end, firstly, we need the concept of block sparsity. In order to emphasize the block structure, similar to [20, 46], we view x as a concatenation of blocks over $\mathcal{I} = \{d_1, d_2, \ldots, d_M\}$. Then x can be expressed as

$$x = (\underbrace{x_1, \ldots, x_{d_1}}_{x[1]}, \underbrace{x_{d_1+1}, \ldots, x_{d_1+d_2}}_{x[2]}, \ldots, \underbrace{x_{N-d_M+1}, \ldots, x_N}_{x[M]})^T \in \mathbb{R}^N,$$

where $x[i]$ denotes the ith block of x with the length d_i and $N = \sum_{i=1}^M d_i$. A vector $x \in \mathbb{R}^N$ is called block $k-$sparse over $\mathcal{I} = \{d_1, d_2, \ldots, d_M\}$ if the number of nonzero vectors $x[i]$ is at most k for $i \in \{1, 2, \ldots, M\}$. Define

$$\|x\|_{2,0} = \sum_{i=1}^{M} I(\|x[i]\|_2 > 0),$$

where $I(\cdot)$ is an indicator function that it equals to 1 if its argument is larger than zero and 0 elsewhere. Then the block $k-$sparse vector over $\mathcal{I} = \{d_1, d_2, \ldots, d_M\}$ can be cast as $\|x\|_{2,0} \le k$. If $d_i = 1$ for all i, block sparsity is just the conventional sparsity. Next, one of the efficient methods to recover block sparse signals is mixed l_2/l_1 minimization

$$\min_x \|x\|_{2,1}, \quad \|y - Ax\|_2 \le \varepsilon, \tag{1.2}$$

where $\|x\|_{2,1} = \sum_{i=1}^M \|x[i]\|_2$. Moreover, mixed norm $\|x\|_{2,2} = (\sum_{i=1}^M \|x[i]\|_2^2)^{1/2}$ and $\|x\|_{2,\infty} = \max_i \|x[i]\|_2$. Note that $\|x\|_{2,2} = \|x\|_2$. It is easy to know the mixed norm minimization is a generalization of conventional norm minimization. To ensure uniqueness and stability of solution for the system (1.1) via mixed l_2/l_1 minimization, Eldar and Mishali [20] generalized the notion of standard restricted isometry property to block sparse vectors, and obtained the following concept of block restricted isometry property (block RIP).