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A POSITIVE AND MONOTONE NUMERICAL SCHEME FOR VOLTERRA-RENEWAL EQUATIONS WITH SPACE FLUXES*

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Abstract

We study a numerical method for solving a system of Volterra-renewal integral equations with space fluxes, that represents the Chapman-Kolmogorov equation for a class of piecewise deterministic stochastic processes. The solution of this equation is related to the time dependent distribution function of the stochastic process and it is a non-negative and non-decreasing function of the space. Based on the Bernstein polynomials, we build up and prove a non-negative and non-decreasing numerical method to solve that equation, with quadratic convergence order in space.

Mathematics subject classification: 65R20, 45D05, 45M20.
Key words: Volterra renewal, Piecewise deterministic process, Monotone positive numerical scheme, Bernstein polynomials.

1. Introduction

In this paper we analyze a numerical method for solving the following system of Volterra-renewal integral equations with space fluxes [6]

$$u_i(x,t) = f_i(x,t) + \sum_{j=1}^{S} q_{ij} \int_0^t k_j(t-\eta) u_j(g_j(x,t,\eta),\eta) d\eta, \tag{1.1}$$

$$\text{where} \qquad f_i(x,t) = \sum_{j=1}^{S} q_{ij} \tilde{F}_j(g_j(x,t,0)) k_j(t), \tag{1.2}$$

for $i = 1, \dots S$, $t \geq 0$ and $x \in \Omega \subset \mathbb{R}$. This system of equations is part of a special form of the Chapman-Kolmogorov equation for a very wide cathegory of stochastic processes named Piecewise Deterministic Processes (PDPs) [13, 14].

Briefly, a PDP is generated from the random switching in time of deterministic motions, taken randomly from a discrete set of given functions. It can be considered as an extension of the "point processes" used in queue theory and renewal processes [27]. From the theoretical side, PDPs are known by experts working in probability calculus and operation research (e.g. see [7, 11, 16]). Within the general category of the PDPs, those characterized by a motion

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switching randomly between deterministic states driven by a semi-Markov process $\mathcal{S}(t)$, are significative. An initial work was made in [22] for Markov processes. A semi-Markov process is a discrete state and continuous time stochastic jump process where the influence of the past is erased at the epochs of jumps. These kind of stochastic processes have potentally a huge amount of applications, we quote Stochastic Hybrid Systems [10, 30] and systems driven by dichotomous noise [26]. Further applications and details of the definition of these PDPs can be found in [5,6], here we give some basic definitions in order to provide a little explanation of the meaning of the terms in Eq. (1.1).

The semi-Markov process is defined as: a discrete Markov process with S states, a stochastic transition matrix $\hat{q} := \{q_{ij}\}$, with $0 \leq q_{ij} \leq 1, \sum_{i=1}^{S} q_{ij} = 1$, jointly to a set of probability density functions $k_i(t) \geq 0, \int_0^\infty k_i(t)dt = 1$, describing the statistics of *switching time events*. The semi-Markov process $\mathcal{S}(t)$ drives the ordinary differential equation

$$dX(t)/dt = \bar{A}_{\mathcal{S}(t)}(X), \tag{1.3}$$

where the function $\bar{A}_i$, is one from a set of $\{\bar{A}_1, \dots, \bar{A}_S\}$ given functions. The resulting motion of the state function $X(t)$ is a random sample path composed of pieces of deterministic trajectories, each of them within two switching events of the semi-Markov process.

The meaningful information of a stochastic process is provided by the marginal probability distribution functions. In this case it is defined as

$$F_i(x, y, t) := \mathbb{P}\Big(X(t) \leq x, y < Y \leq y + dy, i = \mathcal{S}(t)\Big),$$

i.e. the probability that at the time t the process $X(t)$ is in the dynamical state i, for the sojourn time y, and its value is not greater than x. Usually, a probability distribution function is computed by applying Monte Carlo methods directly to the stochastic equation model, like (1.3). This choice is motivated by the easy implementation of the method on computers, but it suffers of a notorious slow convergence rate that scales as the inverse of the square root of the number of samples, although it is robust with respect to the dimension of the spatial domain. Whenever the governing equation of the distribution function is known, it is possible to solve this one by deterministic methods [5, 12], and, if needed, Monte Carlo methods for the validation of theoretical findings (see e.g. [1, 2, 29]). Thus, we search for the probability distribution function by solving the related Chapman-Kolmogorov equation. In the case of PDP described by equation (1.3), the Chapman-Kolmogorov assumes the form of a system of hyperbolic partial differential equations with nonlocal boundary condition [5], or equivalently [6] as the system of Volterra-renewal equation (1.1)-(1.2). Eq. (1.2) is the initial condition of the problem, where $\tilde{F}_j(x)$ represent the distribution functions for the inital data of (1.3). The distribution functions $F_i(x, y, t)$ have the fundamental properties to be monotonically increasing in x and positive in y for all $t > 0$. In order to calculate them, we first solve (1.1)-(1.2), then apply the transformation

$$F_i(x, y, t) = u_i(g_i(x, t, t - y), t - y)\, e^{-\int_0^y \lambda_i(\tau)\, d\tau}, \qquad 0 < y < t, \tag{1.4}$$

where $\lambda_i(t) = k_i(t)/\int_t^\infty k_i(\tau)d\tau$ and the functions $g_j(x, t, \eta)$ represent the inverse fluxes of the solutions of the ODE (1.3). Moreover, if we are interested in the probability distribution without the dependence on the length of the sojourn time y in the state, we integrate it as follows

$$\mathcal{F}_i(x, t) = \int_0^t F_i(x, y, t)\, dy. \tag{1.5}$$