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A CASCADIC MULTIGRID METHOD FOR SEMILINEAR ELLIPTIC EQUATIONS *

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Abstract

This paper introduce a cascadic multigrid method for solving semilinear elliptic equations based on a multilevel correction method. Instead of the common costly way of directly solving semilinear elliptic equation on a very fine space, the new method contains some smoothing steps on a series of multilevel finite element spaces and some solving steps to semilinear elliptic equations on a very coarse space. To prove the efficiency of the new method, we derive two results, one of the optimal convergence rate by choosing the appropriate sequence of finite element spaces and the number of smoothing steps, and the other of the optimal computational work by applying the parallel computing technique. Moreover, the requirement of bounded second order derivatives of nonlinear term in the existing multigrid methods is reduced to a bounded first order derivative in the new method. Some numerical experiments are presented to validate our theoretical analysis.

Mathematics subject classification: 65N30, 65N25, 65L15, 65B99.
Key words: Semilinear elliptic equation, Parallel computing, Cascadic multigrid, Multilevel correction, Finite element method.

1. Introduction

The purpose of this paper is to study the multigrid finite element method for semilinear elliptic problems. As we know, the multigrid methods [4–6, 9, 13, 19, 24] provide optimal order algorithms for solving boundary value problems. The error bounds of the approximate solutions obtained from these efficient numerical algorithms are comparable to the theoretical bounds determined by the finite element discretization. In the past decades, the multigrid method is also applied to nonlinear elliptic problem to improve the efficiency of nonlinear elliptic problem solving, i.e. [19, 25, 26]. In these methods, the Newton iteration is adopted to linearize the nonlinear equations which require bounded second order derivatives of the nonlinear terms. For more information, please refer to [15, 19, 25] and the references cited therein.

Recently, a type of multigrid method for eigenvalue problems has been proposed in [17, 23]. And the corresponding idea can be found in [7, 11, 16]. The aim of this paper is to present a cascadic multigrid method for solving semilinear elliptic equations based on the combination of the multilevel correction method [17, 23] and the cascadic multigrid method for boundary value problems. Similarly to the cascadic multigrid method for the boundary value problem [2, 20],

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we only do the smoothing steps for the involved boundary value problems. Besides, we need to solve some semilinear elliptic equations on a low dimensional space. By organizing suitable numbers of smoothing iteration steps in different levels, the final approximate solution can have the same accuracy as the solution of standard finite element method. In this new version of multigrid method, solving semilinear elliptic problem will not be much more difficult than the multigrid scheme for the corresponding linear boundary value problems. Compared with the existing multigrid method for the semilinear problem, our new method only require a bounded first order derivative of the nonlinear term.

During the numerical calculation, computational complexity and memory consumption increase exponentially with the growth of the scale. As we know, distributed parallel computing can balance the load on each computing node, which will play an important role in the simulation of large scale systems. So as to improve the computational efficiency, we will use the parallel technique to design an algorithm with good scalability.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the semilinear elliptic equation as well as some important properties. The Section 3 is the main part of the paper, where a type of cascadic multigrid algorithm for solving the semilinear elliptic equation and the corresponding error estimate are given. In Section 4, we add the parallel technique to the cascadic multigrid algorithm in Section 3 and estimate the computational work for the parallel algorithm. Four numerical examples are presented in Section 5 to validate our theoretical analysis. Some concluding remarks are given in the last section.

2. Finite Element Method for Semilinear Elliptic Equation

In this paper, the letter C (with or without subscripts) is used to denote a constant which may be different at different places. For convenience, the symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$. Let $\Omega \subset \mathcal{R}^d$ $(d = 2, 3)$ denote a bounded convex domain with Lipschitz boundary $\partial\Omega$. We use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g, [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace. For simplicity, we use $\|\cdot\|_s$ to denote $\|\cdot\|_{s,2,\Omega}$ and V to denote $H_0^1(\Omega)$ in the rest of the paper.

We consider the following type of semilinear elliptic equation:

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) + f(x,u) = g, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A} = (a_{i,j})_{d\times d}$ is a symmetric positive definite matrix with $a_{i,j} \in W^{1,\infty}$ $(i,j = 1,2,\cdots,d)$, and $f(x,u)$ is a nonlinear function corresponding to the second variable and satisfies the following assumption.

Assumption A: The nonlinear function $f(x,\cdot)$ has a non-negative derivative in the second argument

$$0 \leq \frac{\partial f}{\partial v}(x,v) \leq C_f, \quad \forall x \in \Omega \quad \text{and} \quad \forall v \in V. \tag{2.2}$$

The weak form of the semilinear problem (2.1) can be described as: Find $u \in V$ such that

$$a(u,v) + (f(x,u),v) = (g,v), \quad \forall v \in V, \tag{2.3}$$