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AN OVER-PENALIZED WEAK GALERKIN METHOD FOR SECOND-ORDER ELLIPTIC PROBLEMS*

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Abstract

The weak Galerkin (WG) finite element method was first introduced by Wang and Ye for solving second order elliptic equations, with the use of weak functions and their weak gradients. The basis function spaces depend on different combinations of polynomial spaces in the interior subdomains and edges of elements, which makes the WG methods flexible and robust in many applications. Different from the definition of jump in discontinuous Galerkin (DG) methods, we can define a new weaker jump from weak functions defined on edges. Those functions have double values on the interior edges shared by two elements rather than a limit of functions defined in an element tending to its edge. Naturally, the weak jump comes from the difference between two weak functions defined on the same edge. We introduce an over-penalized weak Galerkin (OPWG) method, which has two sets of edge-wise and element-wise shape functions, and adds a penalty term to control weak jumps on the interior edges. Furthermore, optimal *a priori* error estimates in H^1 and L^2 norms are established for the finite element $(\mathbb{P}_k(K), \mathbb{P}_k(e), RT_k(K))$. In addition, some numerical experiments are given to validate theoretical results, and an incomplete LU decomposition has been used as a preconditioner to reduce iterations from the GMRES, CG, and BICGSTAB iterative methods.

Mathematics subject classification: 65N15, 65N30.
Key words: Weak Galerkin, Over-penalized, Finite element methods, Second-order elliptic equation.

1. Introduction

A weak Galerkin (WG) method was first introduced by Wang and Ye [1] for solving general second order elliptic equations, and a series of related numerical analysis and numerical applications to the method are conducted in Ref. [2], which show the WG method efficient and reliable in scientific computing. In general, the WG method refers to a finite element method where differential operators can be approximated by the linear space of vector polynomial functions. The original WG schemes include the polynomial combination $(\mathbb{P}_k(K), \mathbb{P}_k(e), RT_k(K))$ and $(\mathbb{P}_k(K), \mathbb{P}_{k+1}(e), [\mathbb{P}_{k+1}(K)]^d)$ for $k \geq 0$, where $RT_k(K)$ represents the kth order Raviart-Thomas elements [3], $[\mathbb{P}_k(K)]^d$ is a set of polynomials of order no more than k, and d is a dimension of space. To get flexible basis functions and to maintain some kind of weak continuity, Wang and Ye [4, 5] add a stabilizer in variational forms of PDEs. Moreover, WG has been developed for solving more applications, such as elliptic interface problems, Stokes, Helmholtz, Maxwell, etc [6–10].

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For second-order elliptic problems, the weak functions in weak finite element spaces are expressed in a form of $v = (v_0, v_b)$ with v_0 representing the value of v in the interior of each element and v_b on the edges of each element. Generally, polynomial combination $\mathbb{P}_l(K) \times \mathbb{P}_m(e)$ were chosen to be weak finite element spaces, where e denotes the edges or faces of element K, l and m are non-negative integers. The approximation spaces $RT_k(K)$ or $[\mathbb{P}_k(K)]^d$ are chosen for weak differential operators. As far as we know, all WG schemes are based on the fact that weak function v_b along every interior edge is single-valued, however, in this work, we consider the function v_b double-valued on interior edges and for every element and its corresponding edges, the pair of functions (v_0, v_b) are separately defined well.

Due to the treatment of the jumps appearing in the DG methods, we introduce an over-penalized weak Galerkin (OPWG) method for second-order elliptic problems by using a new stabilized term of weak jumps. In other words, shape functions have two traces along each interior edge shared by two neighboring elements, where a weak jump could be generated. Different from the definition of jump in interior-penalty discontinuous Galerkin (IPDG) [11] finite element methods, weak jump comes from the weak functions rather than a limit passing from an interior domain to its edges. We introduce a penalty on the weak jumps, characterizing a new WG method and strengthening the stability and analysis. Therefore, we can also present a new DG method with the use of the definitions of weak functions, because the functions have discontinuity just on the interior edges.

Our main idea is to connect WG with DG methods, and investigate the possibility of penalized methods. In the present work, we do not modify the definition of weak gradient. To change the definition of weak gradient, the reader is referred to a modified WG method [12], in which Wang and Malluwawadu developed a new weak gradient operator defined on piecewise polynomial spaces. We keep the weak finite element spaces and weak gradient operator unchanged except that the shape functions along the interior edges are double-valued. Therefore, many primary results about the WG methods developed before can be easily applied to the present penalized WG method. Furthermore, due to the complete independence of elementwises shape functions similar as in DG, OPWG seems more convenient in parallel computing than the WG methods.

For the sake of simple and easy presentation of the new method, we consider the following second order elliptic equation with nonhomogeneous Dirichlet boundary condition:

$$-\nabla \cdot A \nabla u = f, \qquad \text{in } \Omega, \tag{1.1}$$

$$u = g, \qquad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$, $f \in L^2(\Omega)$ and A is a symmetric and positive definite matrix-valued function in Ω, i.e., there exist two positive numbers $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \xi^t \xi \leq \xi^t A \xi \leq \lambda_2 \xi^t \xi, \qquad \forall \xi \in \mathbb{R}^d,$$

where ξ is a column vector and ξ^t means the transpose of ξ.

The paper is organized as follows. In Section 2, we give some preliminary notations and definitions. In Section 3, the OPWG scheme was introduced. In Section 4, optimal error analysis in H^1 and L^2 norms is established. In Section 5, numerical experiments are conducted to confirm the theoretical results. Some conclusions and remarks are given in the final section.