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HIGH ORDER STABLE MULTI-DOMAIN HYBRID RKDG AND WENO-FD METHODS*

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Abstract

Recently, a kind of high order hybrid methods based on Runge-Kutta discontinuous Galerkin (RKDG) method and weighted essentially non-oscillatory finite difference (WENO-FD) scheme was proposed. Those methods are computationally efficient, however stable problems might sometimes be encountered in practical applications. In this work, we first analyze the linear stabilities of those methods based on the Heuristic theory. We find that the conservative hybrid method is linearly unstable if the numerical flux at the coupling interface is chosen to be 'downstream'. Then we introduce two ways of healing this defect. One is to choose the numerical flux at the coupling interface to be 'upstream'. The other is to employ a slope limiter function to enforce the hybrid method satisfying the local total variation diminishing (TVD) condition. In the end, numerical experiments are provided to validate the effectiveness of the proposed methods.

Mathematics subject classification: 65M60, 65M99, 35L65.
Key words: Runge-Kutta discontinuous Galerkin method, Weighted essentially non-oscillatory scheme, Multi-domain hybrid method, Conservation laws, Heuristic theory.

1. Introduction

In recent years, high order numerical methods have been widely used in solving hyperbolic conservation laws. Among these methods, the weighted essentially non-oscillatory finite volume (WENO-FV) schemes, weighted essentially non-oscillatory finite difference (WENO-FD) schemes, and the Runge-Kutta discontinuous Galerkin finite element (RKDG) methods are often preferred in practical calculations [1]. Here we briefly summarize features of these three methods.

Weighted essentially non-oscillatory (WENO) schemes [2–7] are improvements of the successful essentially non-oscillatory (ENO) reconstruction idea which was originally proposed by Harten [8]. There are two kinds of WENO schemes: one is the finite volume version (WENO-FV) which recovers numerical flux from cell averages, and the other is the finite difference version (WENO-FD) which recovers numerical flux through point values of the solution. Each of the two versions has its advantage and disadvantage. The WENO-FV scheme can handle unstructured meshes, however, its implementation is much more involved and costly for multi-dimensional problems than the WENO-FD scheme. And the WENO-FD scheme is highly

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efficient and easy to achieve high order accuracy, but it has difficulty in handling complex geometries.

As a representative of high order numerical methods, discontinuous Galerkin (DG) methods have become very active in recent years. The first DG method was introduced by Reed and Hill in 1973 [9]. The major development of DG methods was carried out by Cockburn and Shu in a series of papers [10–13]. In their papers, a series of Runge-Kutta local projection discontinuous Galerkin finite element (RKDG) methods were established to solve non-linear time-dependant conservation laws. The RKDG schemes have several important advantages. First, the RKDG methods are better suited than finite difference methods to handle complicated geometries [14,15]. Second, the method can easily handle adaptivity strategies since the refining or un-refining of the grid can be done without taking into account the continuity restrictions typical of conforming finite element methods. Third, the method is highly parallelizable [16]. However, this method also have some issues and weakness, such as huge memory requirements and high computational costs [17].

In order to take advantage of both RKDG methods and WENO schemes, Cheng, Lu and Liu presented multi-domain hybrid RKDG and WENO methods based on computational domain decomposition, in which RKDG methods are employed in the regions near physical boundaries and WENO schemes are applied in other regions [18–20]. They constructed and tested the conservative and non-conservative hybrid methods based on a third order RKDG method with a fifth order WENO-FD scheme (RKDG+WENO-FD). The results showed that the non-conservative hybrid solver has a high order of accuracy in smooth solution and can tackle discontinuities such as shock waves very well due to special strategies applied at the coupling interface. Nevertheless, recent numerical experiments also showed that the method is not unconditionally stable, stability of the conservative hybrid method in some degree depends on the choice of numerical flux at the interface. Our main goals of this work is to analyze the causes of instability of those hybrid methods and then propose ways of suppressing this instability and lead to the development of two versions of high order stable hybrid RKDG and WENO-FD methods.

One version is to choose the 'upstream' numerical flux at the coupling interface. To understand why this strategy works, we first give the necessary stability conditions for the hybrid RKDG and WENO-FD methods by the Heuristic stability theorem [21,22]. The Heuristic stability theorem tells that for a given finite-difference analogue of the partial differential equation (1.1), the numerical scheme is stable

$$u_t + \alpha u_x = 0, \tag{1.1}$$

if the Taylor series expansion of the numerical scheme (1.2)

$$u_t + \alpha u_x = \sum_{n=1}^{\infty} C_{2n}\frac{\partial^{2n} u}{\partial x^{2n}} + C_{2n+1}\frac{\partial^{2n+1} u}{\partial x^{2n+1}}, \tag{1.2}$$

satisfies $(-1)^{r+1}C_{2r} \geq 0$, where

$$2r = \min_{\substack{C_{2n}\neq 0\\ 1\leq n\leq\infty}} 2n$$

is the lowest order of even-order-derivatives whose coefficients are non-zero, otherwise the scheme is not stable. Based on this necessary stability condition, we shall give the 'upstream' strategy of stabilizing a hybrid RKDG and WENO-FD method.

The other version is done based on a lemma of Harten's total variation diminishing (TVD) theorem [23], which tells us that a seme-discrete scheme for (1.1) is TVD and thus stable if it