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THE RECONSTRUCTION OF OBSTACLES IN A WAVEGUIDE USING FINITE ELEMENTS*

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Abstract

This paper concerns the reconstruction of a penetrable obstacle embedded in a waveguide using the scattered data due to point sources, which is formulated as an optimization problem. We propose a fast reconstruction method based on a carefully designed finite element scheme for the direct scattering problem. The method has several merits: 1) the linear sampling method is used to quickly obtain a good initial guess; 2) finite Fourier series are used to approximate the boundary of the obstacle, which is decoupled from the boundary used by the finite element method; and 3) the mesh is fixed and hence the stiffness matrix, mass matrix, and right hand side are assembled once and only minor changes are made at each iteration. The effectiveness of the proposed method is demonstrated by numerical examples.

Mathematics subject classification: 78A46, 65M32, 65M60.
Key words: Inverse scattering problem, Waveguides, Finite element method.

1. Introduction

There are only a few works in literature devoted to the reconstruction of obstacles embedded in periodic structures. In [18], Xu et al. applied a method using generalized dual space indicator for an obstacle in a shallow water waveguide. Dediu and McLaughlin [8] proposed an eigensystem decomposition to recover weak inhomogeneities in a planar waveguide from far-field data. In [4, 5], Bourgeois and Lunéville employed the linear sampling method to reconstruct sound soft obstacles as well as cracks in a planar waveguide from near-field data. A factorization method is used for the inverse scattering problems in a 3D planar waveguide by Arens et al. [1]. For the reconstruction of Dirichlet and impedance obstacles, reverse time migration was employed [6] by Chen and Huang. Recently, a direct sampling method and a multilevel method were introduced in [12, 13] to reconstruct a penetrable inhomogeneous medium embedded in a 3D waveguide [14] and in the stratified ocean waveguide [15], respectively. Note that [4] contains a uniqueness result that the scattered waves correspond to infinitely many incident fields uniquely determine a sound-soft obstacle.

We note that, for a non-planar waveguide, the problem becomes much harder due to the lack of efficient numerical methods for the direct scattering problems. In [17], based on the recursive doubling procedure for periodic structures [9], Sun and Zheng employed the linear sampling method to reconstruct sound soft obstacles using near field data. A similar treatment, but using a different numerical technique for the scattering problem, can be found in [3].

In this paper, we propose an efficient finite element optimization method to reconstruct a penetrable obstacle embedded in a planar waveguide. Consider the waveguide given by

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$\Omega = \mathbb{R}\times[0,H]$, $H>0$. The lower and upper boundaries are $\Sigma_- = \mathbb{R}\times\{0\}$ and $\Sigma_+ = \mathbb{R}\times\{H\}$, which are assumed to be sound-hard. Let D be a star-shaped domain with Lipschitz continuous boundary ∂D. Then there exists a point (z_1, z_2) and a periodic function r such that the boundary ∂D can be represented as

$$\partial D := \{(z_1, z_2) + r(t)(\cos t, \sin t) | \, 0 \leqslant t < 2\pi\}.$$

The inverse problem is to determine the location and shape of D from the measured scattered field due to incident waves by point sources. In particular, the point sources locate on Γ_i and the scattered field is measured on Γ_m, where Γ_i and Γ_m are line segments (see Figure 1.1). For example, $\Gamma_i = \Gamma_m = \Gamma_1 \cup \Gamma_2$.

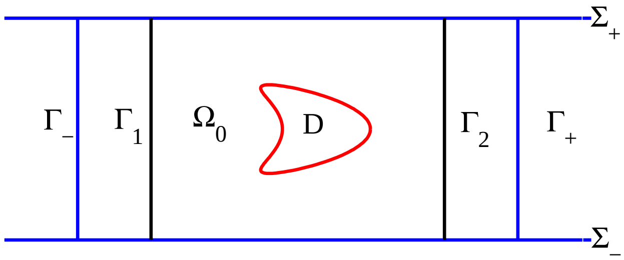

Fig. 1.1. The physical configuration of the scattering problem.

We assume that r can be represented by a finite Fourier series. The inverse problem is formulated as an optimization problem, which is solved by a quasi-Newton method. The initial guess is obtained using the linear sampling method [17]. Since a lot of direct scattering problems need to be solved numerically, we design an efficient finite element method using a fixed mesh (see [19]). The stiffness matrix, mass matrix, and right-hand side only need to be assembled once at the beginning of the numerical procedure. Only minor changes are necessary in the subsequent iterations. Multiple frequency data are used to obtain a better reconstruction (see, e.g., [2,16,19]).

The rest of paper is organized as follows. Section 2 introduces the direct scattering problem. In Section 3, the inverse problem is formulated as an optimization problem, and the Fréchet derivative of the target function is studied. In Section 4, we present the reconstruction method in detail. Section 5 contains numerical results.

2. The Direct Problem

Let $k = w/c$ be the wavenumber, where w is the frequency and c is the speed of sound. For planar waveguides, there exist eigenvalues and eigenfunctions of Sturm-Liouville type (see [8]) given by

$$k_n = \frac{n\pi}{H}, \quad \theta_n(x_2) = \begin{cases} \sqrt{\frac{1}{H}}, & n = 0, \\ \sqrt{\frac{2}{H}}\cos\left(\frac{n\pi}{H}x_2\right), & n \geqslant 1. \end{cases}$$

The incident field $u^i := G(\cdot, y)$, generated by a point source located on Γ_i, is defined as (see [4])

$$G(x,y) = \sum_{n\in\mathbb{N}} \frac{e^{i\beta_n|x_1-y_1|}}{2i\beta_n}\theta_n(x_2)\theta_n(y_2),$$