Journal of Computational Mathematics
Vol.36, No.1, 2018, 128–158.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1612-m2016-0495

FAST SPECTRAL GALERKIN METHOD FOR LOGARITHMIC SINGULAR EQUATIONS ON A SEGMENT*

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Abstract

We present a fast Galerkin spectral method to solve logarithmic singular equations on segments. The proposed method uses weighted first-kind Chebyshev polynomials. Convergence rates of several orders are obtained for fractional Sobolev spaces $\widetilde{H}^{-1/2}$ (or $H_{00}^{-1/2}$). Main tools are the approximation properties of the discretization basis, the construction of a suitable Hilbert scale for weighted L^2-spaces and local regularity estimates. Numerical experiments are provided to validate our claims.

Mathematics subject classification: 65R20, 65F35, 65N22, 65N38.
Key words: Screen problems, Boundary integral operators, Spectral methods.

1. Introduction

We study elliptic problems in $\mathbb{R}^2$ having as common ground unbounded domains with cuts. Such domains are not even Lipschitz and usually fall in the category of screen, crack or interface problems [8,27,28,33]. Our focus lies on the analysis of integral logarithmic singular operators appearing in the associated integral representations. In the simplest scenario, let $\Gamma_c \subset \mathbb{R}^2$ be an open Jordan curve, $\Omega := \mathbb{R}^2 \setminus \overline{\Gamma}_c$ an isotropic medium, and consider the following problem: find u such that

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega, \\ u = g & \text{for} \quad \mathbf{x} \in \Gamma_c, \end{cases} \tag{1.1}$$

where g is a given datum in a suitable functional space. It is well known [22], that the potential u can be represented as the *single layer potential*:

$$u(\mathbf{x}) = \int_{\Gamma_c} \log \frac{1}{\|\mathbf{x}-\mathbf{y}\|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y}, \qquad \text{for} \quad \mathbf{x} \in \Omega, \tag{1.2}$$

a representation known as the *indirect method* [30], wherein φ is the solution of the logarithmic singular integral equation:

$$g(\mathbf{x}) = \int_{\Gamma_c} \log \frac{1}{\|\mathbf{x}-\mathbf{y}\|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y}, \qquad \text{for} \quad \mathbf{x} \in \Gamma_c. \tag{1.3}$$

* Received January 4, 2016 / Accepted December 5, 2016 /
Published online October 11, 2017 /

This type of first-kind Fredholm equation has received considerable attention in the past [2, 10, 14, 19, 31] with several strategies put forward to solve it numerically.

More recently, explicit expressions for the inverse of the weakly singular and hypersingular operators over a straight segment $(-1,1)$ as well as Calderón-type identities were provided in [17]. These results were computationally implemented and numerically analyzed for non-uniform low-order discretization in [16]. Still, the number of degrees of freedom required to attain high accuracy remains impractical when performing large scale/numbers of simulations. For example, this is the case when the Monte Carlo method is used to compute statistical moments of the screen problem solutions subject to uncertainties in either geometrical features or sources. As an alternative remedy, one can use p-refinement schemes, sometimes referred to as *spectral methods* [3, 4, 7]. In [13], the authors provide convergence results for polygonal segments. Along both lines of work, Lintner and Bruno [5, 6] developed a generalized Calderón formula for open arcs. When combined with their high-order Nyström methods, they observe excellent performance of their Calderón preconditioner for a wide range of geometries and wave propagation problems. However, no mathematical analysis of their method is available.

In this work, we provide a fast spectral Galerkin method to solve general logarithmic singular kernels. Although not a new idea for structures of co-dimension one in $\mathbb{R}^2$ [14, 31] in the setting of weighted Sobolev spaces, the generalization to classic Sobolev spaces and interface problems seems rather original. Without loss of generality, we will remit to Γ_c given by bounded Jordan arcs, the case of semi-infinite or infinite cuts not being considered. By changing coordinates, the integral equation is cast over the canonical interval $(-1,1)$, so that it has the form of a compactly perturbed logarithmic integral operator. As we will show, solutions are characterized by square-root singularities at the endpoints $\{\pm 1\}$ and weighted Chebyshev polynomials naturally define a basis for numerical approximation. Though we had a different aim, we obtain similar results to those published by Dominguez *et al.* [15] wherein so-called *Hilbert scales* are studied along with their connection to weighted Sobolev spaces.

The remaining of this work is organized as follows. In Section 2, we recall certain definitions and introduce notation conventions. Section 2.6 recalls uniqueness and existence results [17] for the exterior Dirichlet problem for the Laplacian in $\mathbb{R}^2$ with a line segment removed that lead to (1.3). After presenting Sobolev spaces over $[0,\pi]$ and weighted L^2-spaces in Sections 3 and 4, we establish convergence results and truncation errors for associated series expansions in Section 5. Although many of the presented results are well known [10, 21, 23], we will emphasize the link between trigonometric and Chebyshev polynomials as well as their profound connection with logarithmic operators. In this sense, we extend previous results obtained in the setting of Hölder continuous functions developed in [24, 25]. Numerical results are discussed in Section 6 and conclusions drawn in Section 7.

2. Preliminaries

2.1. Model geometry

We start by recalling the canonic splitting of $\mathbb{R}^2$ into two half-planes:

$$\pi_{\pm} := \left\{ \mathbf{x} = (x,y) \in \mathbb{R}^2 \,:\, y \lesseqgtr 0,\ \forall\, y \in \mathbb{R} \right\}, \tag{2.1}$$

with interface Γ given by the line $y = 0$. The interface is further divided into the open disjoint segments $\Gamma_c := (a_-, a_+) \times \{0\}$, where $a_-, a_+ \in \mathbb{R}$ are such that $-\infty < a_- < a_+ < \infty$, and