Journal of Computational Mathematics
Vol.35, No.6, 2017, 828–838.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1701-m2015-0443

RECURSIVE INTEGRAL METHOD FOR THE NONLINEAR NON-SELFADJOINT TRANSMISSION EIGENVALUE PROBLEM*

Yingxia Xi and Xia Ji

LSEC, NCMIS, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email: yxiaxi@lsec.cc.ac.cn, jixia@lsec.cc.ac.cn

Abstract

The transmission eigenvalue problem is an eigenvalue problem that arises in the scattering of time-harmonic waves by an inhomogeneous medium of compact support. Based on a fourth order formulation, the transmission eigenvalue problem is discretized by the Morley element. For the resulting quadratic eigenvalue problem, a recursive integral method is used to compute real and complex eigenvalues in prescribed regions in the complex plane. Numerical examples are presented to demonstrate the effectiveness of the proposed method.

Mathematics subject classification: 34L16, 65L60.
Key words: Transmission eigenvalue problem, Nonlinear eigenvalue problem, Contour integrals.

1. Introduction

The transmission eigenvalue (TE) problem has important applications in inverse scattering theory and has attracted attention of many researchers recently [3, 4, 7–9, 25]. Although stated in a simple form, the TE problem is nonstandard and not covered by classical theories.

In this paper, we consider the numerical treatment of the TE problem. The first effort was given in [8], where three finite element methods (FEM) were proposed. Two iteration methods were proposed with rigorous convergence analysis in [23]. A mixed method was developed in [13]. An efficient spectral element based numerical method for the 2D TE problem of radially stratified media was given in [1]. Recently, a mixed FEM with convergence proof was proposed to solve the TE problem in [5], and another FEM based method, which transforms the TE problem into a quadratic eigenvalue problem (QEP), was given in [19]. The related source problem [10, 26] and other multilevel [12] and multigrid [14] type methods have also been discussed.

Some non-traditional methods, such as the linear sampling method [24] and the inside-out duality method [18], have been applied to estimate TEs using scattering data. However, these methods need to solve a tremendous number of direct problems which make them computationally prohibitive.

Recently, contour integral based methods have been successfully applied to solve eigenvalue problems. In [21, 22], a contour integral based method was proposed to compute eigenvalues of a generalized eigenvalue problem, which lie in a specific region in the complex plane. And in [2], a contour integral based method was used to solve a nonlinear eigenvalue problem (NEP)

* Received November 4, 2015 / Revised version received October 7, 2016 / Accepted January 27, 2017 / Published online September 13, 2017 /

by formulating the NEP as a linear eigenvalue problem (LEP). In [17], a contour integral based method was used to compute TEs in a special case when the index of refraction is a constant. In [11], a novel recursive integral method (RIM) was proposed to solve LEPs. Based on the eigenprojections of compact operators, regions are searched recursively to test whether there exist eigenvalues lying inside a specific contour in the complex plane. RIM is effective, robust, and essentially parallelizable. According to [11], even for the case when the matrix is non-Hermitian and degenerate, and the information of the spectrum structure is unknown, RIM still works successfully.

This paper focuses on the extension of RIM to solve NEPs and numerically shows its robustness. We derive the formulation of RIM for a QEP. Both real and complex eigenvalues are computed effectively. We discretize the TE problem by the nonconforming Morley element. For the derived QEP, we employ RIM to find both real and complex eigenvalues within a specific contour in the complex plane. The TE problem is indeed a fourth order problem [4]. Writing it in a mixed form as in [11] would require higher regularity of eigenfunctions which in turn leads to a restriction of the domain, see, e.g., Chp. 4 of [25]. It is known that a similar mixed finite element approach for the biharmonic equation leads to spurious eigenvalues, which could happen when the domain is non-convex. Hence it is preferable to discretize the weak formulation directly and the Morley element is a suitable non-conforming element for triangular meshes. We only provide a theoretical proof for the Morley element when the refractive index $n(x)$ is a constant [15], while the proof for more general case when $n(x)$ is non-constant is still open.

This paper is organized as follows. In Section 2, we introduce the TE problem, the reduced fourth order formulation and the corresponding nonconforming FEM discretization. In Section 3, RIM for the QEP is given. In Section 4, numercial examples are presented to validate the algorithm. Section 5 is the conclusion part.

2. The Transmission Eigenvalue Problem

2.1. Formulation

For the scattering of time-harmonic acoustic waves by a bounded and simply connected inhomogeneous medium $\Omega \subset \mathcal{R}^2$, the transmission eigenvalue problem is to find $k \in \mathcal{C}$ and $\phi, \varphi \in H^2(\Omega)$ such that

$$\begin{cases} \Delta\phi + k^2 n(x)\phi = 0, & \text{in } \Omega, \\ \Delta\varphi + k^2\varphi = 0, & \text{in } \Omega, \\ \phi - \varphi = 0, & \text{on } \partial\Omega, \\ \frac{\partial\phi}{\partial\nu} - \frac{\partial\varphi}{\partial\nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ν is the unit outward normal to the boundary $\partial\Omega$. The index of refraction $n(x)$ is either $n(x) > \alpha_0$ a.e. in Ω for some constant $\alpha_0 > 1$, or $0 < n(x) < \tilde{\alpha}_0$ a.e. in Ω for some constant $\tilde{\alpha}_0 < 1$. Here we consider the first case. The second one follows similarly. Complex number k for which there exists a nontrivial solution to (2.1) is called a TE [8].

We define

$$V := H_0^2(\Omega) = \left\{ u \in H^2(\Omega) :\ u = 0 \text{ and } \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega \right\},$$

and denote (u, v) the $L^2(\Omega)$ inner product. Introducing a new variable $u = \phi - \varphi \in V$ and following the same procedure in [13], one finds that u and k satisfy the fourth order problem

$$\big(\Delta + k^2 n(x)\big)\frac{1}{n(x) - 1}(\Delta + k^2)u = 0. \tag{2.2}$$