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EXPONENTIALLY FITTED TRAPEZOIDAL SCHEME FOR A STOCHASTIC OSCILLATOR*

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Abstract

This paper applies exponentially fitted trapezoidal scheme to a stochastic oscillator. The scheme is convergent with mean-square order 1 and symplectic. Its numerical solution oscillates and the second moment increases linearly with time. The numerical example verifies the analysis of the scheme.

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Key words: Exponentially fitted trapezoidal scheme, Symplectic; mean-square order, Second moment.

1. Introduction

Stochastic oscillators are important mathematical models to evaluate the advantages and disadvantages of numerical schemes ([1–4]). Many papers discuss the linear stochastic oscillator $\ddot{u}(t)+u(t)=\lambda\dot{W}(t)$, equivalently,

$$\begin{cases} du(t)=v(t)dt, \\ dv(t)=-u(t)dt+\lambda dW(t), \\ u(0)=1, \quad v(0)=0, \end{cases} \tag{1.1}$$

where λ is given and $W(t)$ is Brownian motion.

Proposition 1.1. ([3,4]) *The Hamiltonian stochastic system* (1.1) *preserves the symplectic* 2-*form* $du(t)\wedge dv(t)$. *Its solution can be expressed as*

$$u(t)=\cos(t)+\lambda\int_0^t \sin(t-s)dW(s), \tag{1.2a}$$

$$v(t)=-\sin(t)+\lambda\int_0^t \cos(t-s)dW(s). \tag{1.2b}$$

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$u(t)$ oscillates with infinitely many simple zeros. Moreover, it satisfies that

$$E(u(t)^2 + v(t)^2) = 1 + \lambda^2 t.$$

The proposition above tells us the solution phase flow of (1.1) is symplectic, its second moment grows linearly with time and its solution oscillates. Many papers discuss the symplectic and multi-symplectic schemes ([5–10]). The references show that numerical preservation of symplectic structure is as important as high accuracy for numerical solutions to Hamiltonian systems. Symplectic schemes can simulate stably the main qualitative property of solutions of Hamiltonian systems for long time.

Recently, exponentially fitted schemes are popular to solve ODE systems with oscillating solutions ([11–14]). They can be derived by generalizing the Runge-Kutta methods and choosing the coefficients in order to integrate exactly all functions in a selected linear space and minimize the local error of the numerical solution. [11] proposes exponentially fitted Runge-Kutta methods with s stages and introduces how to estimate the parameter. An explicit exponentially fitted Runge-Kutta method with an optimal parameter is presented in [12]. Exponentially fitted Runge-Kutta-Nyström method is constructed for second order ODE systems with oscillating solutions [13]. [14] studies two kinds of exponentially fitted Runge-Kutta methods with fixed points and with frequency-dependent points. For Hamiltonian systems, exponentially fitted schemes with structure preservation properties are proposed in [15, 16, 17, 18]. [17] proposes a fourth-order symplectic exponentially fitted scheme with modified Gauss method. [18] considers linear and quadratic discrete invariants of exponentially fitted Runge-Kutta methods.

In [4], the predictor-corrector methods are applied to the stochastic oscillator system (1.1). This work inspires us some idea. How does exponentially fitted scheme behave for the stochastic oscillator system (1.1)? Whether is the numerical solution symplectic and oscillatory? Whether does the numerical second moment grow linearly with time? In this paper we try to study and answer the question.

2. Exponentially Fitted Trapezoidal Scheme

Two stage exponentially fitted Runge-Kutta scheme with symmetric nodes applied to deterministic first-order system $y' = f(x, y)$ yields that

$$\begin{aligned} y_1 &= \gamma_0 y_0 + h(b_1 f(x_0 + c_1 h, Y_1) + b_2 f(x_0 + c_2 h, Y_2)), \\ Y_1 &= \gamma_1 y_0 + h(a_{11} f(x_0 + c_1 h, Y_1) + a_{12} f(x_0 + c_2 h, Y_2)), \\ Y_2 &= \gamma_2 y_0 + h(a_{21} f(x_0 + c_1 h, Y_1) + a_{22} f(x_0 + c_2 h, Y_2)). \end{aligned} \tag{2.1}$$

Here $c_1 = 1/2 - d, c_2 = 1/2 + d$. To make the scheme exact for the linear fitting trigonometric space generated by $\{1, exp(\pm i\omega x)\}$ and symplectic [18], we get exponentially fitted trapezoidal scheme with $d = 1/2, \gamma_0 = \gamma_1 = \gamma_2 = 1, a_{11} = a_{12} = 0, a_{21} = a_{22} = b_1 = b_2 = \frac{1-\cos(\omega h)}{\omega h \sin(\omega h)}$, equivalently,

$$y_1 = y_0 + \frac{1 - \cos(\omega h)}{\omega \sin(\omega h)} \Big(f(x_0, y_0) + f(x_1, y_1) \Big). \tag{2.2}$$

Taylor expansion yields that the local truncation error is $(y^{(3)} + \omega^2 y')h^3/12$. Therefore, the scheme is convergent with order 2. If ω is chosen such that $y^{(3)} + \omega^2 y' = 0$, then the scheme will be convergent with order 3 [12]. Under the segmentation with equidistant points, we apply