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LOCAL STRUCTURE-PRESERVING ALGORITHMS FOR THE KDV EQUATION*

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Abstract

In this paper, based on the concatenating method, we present a unified framework to construct a series of local structure-preserving algorithms for the Korteweg-de Vries (KdV) equation, including eight multi-symplectic algorithms, eight local energy-conserving algorithms and eight local momentum-conserving algorithms. Among these algorithms, some have been discussed and widely used while the most are new. The outstanding advantage of these proposed algorithms is that they conserve the local structures in any time-space region exactly. Therefore, the local structure-preserving algorithms overcome the restriction of global structure-preserving algorithms on the boundary conditions. Numerical experiments are conducted to show the performance of the proposed methods. Moreover, the unified framework can be easily applied to many other equations.

Mathematics subject classification: 65L12, 65M06, 65M12.
Key words: Korteweg-de Vries (KdV) equation, structure-preserving algorithms, concatenating method, multi-symplectic conservation law.

1. Introduction

Consider the system of Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + \eta u \frac{\partial u}{\partial x} + \mu^2 \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, \tag{1.1}$$

where η and μ are both real constants. It is an important nonlinear hyperbolic equation with smooth solutions at all times and also a mathematical model of waves on shallow water surfaces [1]. Eq. (1.1) has been used to describe various kinds of phenomena, such as waves in bubble-liquid mixtures, acoustic waves in an anharmonic crystal, magnetohydrodynamic wave in warm plasmas and ion acoustic wave. The KdV equation was originally introduced by Zabusky-Kruskal [2] who discovered the soliton in 1965.

Based on the rule that numerical algorithms should preserve the intrinsic properties of the original problems as much as possible, Feng [3] first presented the concept of symplectic schemes for Hamiltonian systems and further the structure-preserving algorithms for the general conservative dynamical systems. Theoretical analysis and practical computations both prove that

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the symplectic schemes have very wide and significant applications in many fields due to their excellent stability and accurate long time simulations [4–6]. Marsden et al. [7], Bridges [8] and Reich [9] introduced the concept of multi-symplectic structure and multi-symplectic integrator for Hamiltonian partial differential equations (PDEs), which can be regarded as the direct generalization of symplectic integrator. Afterwards, multi-symplectic algorithms developed very fast and a lot of achievements have been obtained. For example, Hong et al. [10] and Liu et al. [11] proposed the multi-symplectic Runge-Kutta methods for nonlinear Dirac equations and Hamiltonian equations, respectively. Sun et al. [12] and Kong et al. [13] investigated the multi-symplectic methods for Maxwell equation. Moreover, Hong et al. [14] studied the multi-symplecticity of partitioned Runge-Kutta methods for Hamiltonian PDEs. Wang and Hong [15] reviewed the development of multi-symplectic algorithms for Hamiltonian PDEs. Actually, besides the geometric structure, the idea of Feng's structure-preserving algorithms also contains other conservative properties of the PDEs, such as the physical conservation laws like energy or momentum conservation law and the algebraic characters. In some fields, it is convenient sometimes to construct numerical algorithms that preserve the physical conservation law rather than the symplectic or multi-symplectic ones.

It is noted that Wang et al. [16] proposed the concept of the local structure-preserving algorithm for PDEs, and then constructed some algorithms preserved the multi-symplectic conservation law, local energy and momentum conservation laws for the Kleign-Gordon equation by using the concatenating method. Cai et al. [17, 18] applied successfully the theory of the local structure-preserving algorithm to the "good" Boussinesq equation and the coupled nonlinear Schrödinger system. The main advantage of local structure-preserving algorithm is that they conserve the local structures of PDEs in any local time-space region. In other words, they can overcome the restriction of global structure-preserving algorithm on the boundary conditions.

On the other hand, there have been many numerical methods for solving the KdV equation, such as finite difference methods, finite element methods, spectral and pseudo-spectral methods. Note that structure-preserving algorithms play an important role in the development of these numerical methods. In particular, a 12-point multi-symplectic scheme for KdV equation was derived by Zhao and Qin [19], and a family of symplectic and multi-symplectic box schemes of KdV equation was investigated by Ascher and McLachlan [20]. Wang et al. [21] discussed an explicit 6-point multi-symplectic scheme which did not show the nonlinear instabilities and unphysical oscillations when used to simulate the collision of multiple solitary wave. In [22], Chen et al. constructed a multi-symplectic Fourier pseudo-spectral method and a multi-symlpectic wavelet collocation method for the Ito-type coupled KdV equation. In [23], Cui et al. developed a finite-volume scheme for the KdV equation, which conserved both the momentum and energy using the operator splitting approach. However, the above existing algorithms are put forward and studied just individually and except [23], there is few energy-conserving or momentum-conserving methods in literatures, which motivates us to study the local structure-preserving algorithms of the KdV equation systematically.

In this paper, we emphasize how to construct a unified framework of the local structure-preserving algorithms for KdV equation by using the concatenating method. This method to construct difference schemes for PDEs is different from the method of lines and the alternating direction method. Its basis idea comes from the Runge-Kutta method which deals with PDEs by space and time separately. In 2000, Reich [24] proved that concatenating the Runge-Kutta method in type Gauss collocation can lead to multi-symplectic schemes for the nonlinear wave equation. In [16–18, 25], Wang et al. used this method to construct a series of multi-symplectic