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LOCAL ANALYSIS OF THE FULLY DISCRETE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE TIME-DEPENDENT SINGULARLY PERTURBED PROBLEM*

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Abstract

In this paper we consider the fully discrete local discontinuous Galerkin method, where the third order explicit Runge-Kutta time marching is coupled. For the one-dimensional time-dependent singularly perturbed problem with a boundary layer, we shall prove that the resulted scheme is not only of good behavior at the local stability, but also has the double-optimal local error estimate. It is to say, the convergence rate is optimal in both space and time, and the width of the cut-off subdomain is also nearly optimal, if the boundary condition at each intermediate stage is given in a proper way. Numerical experiments are also given.

Mathematics subject classification: 65M15, 65M60.
Key words: Local analysis, Runge-Kutta method, Local discontinuous Galerkin method, Singularly perturbed problem, Boundary layer.

1. Introduction

Given the final time $T > 0$ and the bounded domain $I = (a, b)$, consider the singularly perturbed problem in one-dimensional space

$$u_t - \varepsilon u_{xx} + u_x + cu = f(x,t), \quad (x,t) \in I \times (0,T], \tag{1.1a}$$

subject to the initial solution

$$u(x,0) = u_0(x), \quad x \in I, \tag{1.1b}$$

and the Dirichlet boundary condition

$$(u(a,t), u(b,t))^\top = (g_a(t), g_b(t))^\top \equiv \boldsymbol{g}(t), \quad t \in (0,T], \tag{1.1c}$$

where $0 < \varepsilon \ll 1$ and $c \geq 0$ are two constants. Here $f(x,t), u_0(x), g_a(t)$ and $g_b(t)$ are given functions and assumed to be smooth, such that this problem has a sufficiently smooth solution. However, the solution often varies quickly with a huge gradient nearby the outflow boundary point $x = b$, and forms a boundary layer there. This causes numerical difficulties for lots of traditional methods. Hence, this singularly perturbed problem and its numerical study have been paid much attention to in a long period. Till now, many successful algorithms have been presented and developed. For example, there are the standard finite element method combined with the layer-adapted mesh [17, 23], streamline upwinding Petrov-Galerkin method [10, 16],

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interior penalty discontinuous Galerkin (IPDG) method [8,19], the local discontinuous Galerkin (LDG) method [3, 4, 18, 24], and so on, among finite element methods.

The LDG method considered in this paper was introduced firstly by Cockburn and Shu [6] to solve the convection diffusion problem, motivated by the work of Bassi and Rebay [1]. As a special class of discontinuous Galerkin (DG) method, the main technique of the LDG method is to rewrite the considered problem into an equivalent system containing only the first-order derivatives, which can be further discretized by the standard DG method [5, 12]. Since discontinuous finite element spaces do not require continuity at element boundaries, the LDG method has enough flexibility to deal with the fast-varying solutions, and even the discontinuous solutions. For a fairly complete set of references about the method and its implementation, please refer to the review papers [7, 14] and recent book [11].

Compared with the wide applications of DG method, the theoretical conclusions are lack in some sense. Many error estimates to the DG method of problem (1.1) are carried out in the global regularity assumption, namely, the exact solution is assumed to be smooth enough in the whole domain. For example, the L^2-norm error estimate has been considered for the semi-discrete LDG method in [3], and for the fully discrete third order explicit Runge-Kutta time-marching in [18]. However, the obtained result seems useless when the diffusion parameter ε goes to zero. Hence, the local analysis is necessary to show out the numerical advantage of the LDG method. As far as the authors know, there are only couples of work on this issue, for example, Johnson et al. [9] for the (space-time) DG method, and Guzmán [8] for the IPDG method. Recently, Zhu and Zhang [24] considered the LDG method to solve the one-dimensional steady problem. Furthermore, Cheng and Zhang [4] extended the local analysis to the time-dependent problem, and obtained the double-optimal error estimate that the convergence rate in the local L^2-norm is optimal and the width of the cut-off region nearby the outflow boundary is almost optimal, namely, in the order like $\mathcal{O}(h\log(1/h))$, where h is the size of the spatial mesh.

The fully discrete LDG method was also considered in [4]. However, the second order explicit total variation diminishing Runge-Kutta (TVDRK2) time-marching was discussed only. In this paper we shall extend the above work and consider the LDGRK3 scheme where the third order explicit total variation diminishing Runge-Kutta (TVDRK3) time discretization is adopted. The LDGRK3 scheme is more popular in practice, owing to the good stability for piecewise polynomials with arbitrary degree under the standard temporal-spatial condition and higher order accuracy in time [18], in the case that $\varepsilon < h$. Similar as in [4], the main technique used in this paper is the energy technique with a suitable weight function. But, the local analysis of LDGRK3 scheme is more complex than that of LDGRK2 scheme. The reason comes from two issues. Firstly, we have to use fully the anti-symmetry property (see Lemma 3.2) of the LDG discretization in space direction, in order to control each term in the energy equation. Secondly, we have to deal carefully with the boundary condition to avoid the accuracy reduction. Many technical process can be viewed as an extension of the work in [18] and [4]. As a little highlight of this paper, we will consider explicitly the source term $f(x,t)$, which maybe depend on the time explicitly. Since the problem is not autonomous, the different treatment in the TVDRK3 time-marching will lead to some difficulties in the theoretical analysis. To overcome this, we have to modify the definition of the reference functions in [18] where the source term is equal to zero.

The rest of this paper is organized as follows. In Section 2, we present the LDGRK3 scheme for the singularly perturbed problem (1.1). In Section 3, we consider a general framework