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FINITE ELEMENT EXTERIOR CALCULUS FOR EVOLUTION PROBLEMS*

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Abstract

Arnold, Falk, and Winther [*Bull. Amer. Math. Soc.* **47** (2010), 281–354] showed that mixed variational problems, and their numerical approximation by mixed methods, could be most completely understood using the ideas and tools of *Hilbert complexes*. This led to the development of the Finite Element Exterior Calculus (FEEC) for a large class of linear elliptic problems. More recently, Holst and Stern [*Found. Comp. Math.* **12**:3 (2012), 263–293 and 363–387] extended the FEEC framework to semi-linear problems, and to problems containing *variational crimes*, allowing for the analysis and numerical approximation of linear and nonlinear geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing surface finite element approximation theory. In this article, we develop another distinct extension to the FEEC, namely to parabolic and hyperbolic evolution systems, allowing for the treatment of geometric and other evolution problems. Our approach is to combine the recent work on the FEEC for elliptic problems with a classical approach to solving evolution problems via semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in time-parameterized Hilbert spaces (or *Bochner* spaces). Building on classical approaches by Thomée for parabolic problems and Geveci for hyperbolic problems, we establish *a priori* error estimates for Galerkin FEM approximation in the natural parametrized Hilbert space norms. In particular, we recover the results of Thomée and Geveci for two-dimensional domains and lowest-order mixed methods as special cases, effectively extending their results to arbitrary spatial dimension and to an entire family of mixed methods. We also show how the Holst and Stern framework allows for extensions of these results to certain semi-linear evolution problems.

Mathematics subject classification: 65M15, 65M20, 65M60.
Key words: FEEC, Elliptic equations, Evolution equations, Nonlinear equations, Approximation theory, Nonlinear approximation, Inf-sup conditions, *A priori* estimates.

1. Introduction

More than two decades of research on linear mixed variational problems, and their numerical approximation by mixed methods, recently culminated in the seminal work of Arnold, Falk, and

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Winther [3]. The authors show how such problems are most completely understood using the ideas and tools of *Hilbert complexes*, leading to the development of the Finite Element Exterior Calculus (FEEC) for elliptic problems. In two related articles [20,21], Holst and Stern extended the Arnold–Falk–Winther framework to include *variational crimes*, allowing for the analysis and numerical approximation of linear and nonlinear geometric elliptic partial differential equations on Riemannian manifolds of arbitrary spatial dimension, generalizing the existing surface finite element approximation theory in several directions.

In this article, we extend the FEEC in another direction, namely to parabolic and hyperbolic evolution systems. Our approach is to combine the recent work on the FEEC for elliptic problems with a classical approach to solving evolution problems using semi-discrete finite element methods, by viewing solutions to the evolution problem as lying in time-parameterized Banach (or *Bochner*) spaces. Building on classical approaches by Thomée for parabolic problems and Geveci for hyperbolic problems, we establish *a priori* error estimates for Galerkin FEM approximation in the natural parametrized Hilbert space norms. In particular, we recover the results of Thomée and Geveci for two-dimensional domains and the lowest-order mixed method as a special case, effectively extending their results to arbitrary spatial dimension and to an entire family of mixed methods. We also show how the Holst and Stern framework allows for extensions of these results to certain semi-linear evolution problems.

To understand why the finite element exterior calculus (FEEC) has emerged in a natural way to become a major mathematical tool in the development of numerical methods for PDE, we recall one of the many examples presented at length in [3]. Consider the vector Laplacian over a domain $\Omega \subset \mathbb{R}^3$:

$$-\Delta u := -\ \mathrm{grad}\ \ \mathrm{div}\ u + \ \mathrm{curl}\ \ \mathrm{curl}\ u,$$

with the boundary conditions $u \cdot n = 0$ and $\mathrm{curl}\ u \times n = 0$ on $\partial\Omega$. Given an L^2 vector field data f, the natural variational formulation of the problem is: Find $u \in H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div};\Omega)$ such that

$$\int_\Omega [(\nabla \cdot u)(\nabla \cdot v) + (\nabla \times u) \cdot (\nabla \times v)]\ dx = \int_\Omega f \cdot v\ dx,$$
$$\forall v \in H(\mathrm{curl};\Omega) \cap H_0(\mathrm{div};\Omega). \tag{1.1}$$

By introducing an intermediate variable $\sigma := -\mathrm{div}\ u$, a natural alternative formulation is the following *mixed* form: find $(\sigma, u) \in H^1(\Omega) \times H(\mathrm{curl};\Omega)$ such that

$$\int_\Omega (\sigma\tau - u \cdot \nabla\tau)\ dx = 0, \qquad \forall \tau \in H^1(\Omega), \tag{1.2}$$

$$\int_\Omega [\nabla\sigma \cdot v + (\nabla \times u) \cdot (\nabla \times v)]\ dx = \int_\Omega f \cdot v\ dx, \quad \forall v \in H(\mathrm{curl};\Omega). \tag{1.3}$$

Using the standard finite element approach based on the non-mixed formulation (1.1) (e.g., using continuous piecewise linear vector functions) can yield incorrect results if the domain has certain geometric features (e.g., domains with re-entrant corners) or topological features (e.g., non-simply connected domains). A standard finite element approach based on the mixed formulation (1.2)-(1.3), on the other hand, suffers neither of these difficulties and typically works extremely well.

The explanation for why one approach fails and the other succeeds lies in the fundamental mathematical structures underlying the finite element method. In case of the domain with re-entrant corners, the failure of the non-mixed approach is due to the non-density of $H^1(\Omega)$ in