Journal of Computational Mathematics
Vol.35, No.1, 2017, 1–18.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1608-m4507

STABILITY ANALYSIS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH NONLINEAR ABSORBING BOUNDARY CONDITIONS*

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Abstract

Local absorbing boundary conditions (LABCs) for nonlinear Schrödinger equations have been constructed in papers [PRE 78(2008) 026709; and PRE 79 (2009) 046711] using the so-called unified approach. In this paper, we present stability analysis for the reduced problem with LABCs on the bounded computational domain by the energy estimate, and discuss a class of modified versions of LABCs. To prove the stability analysis of the reduced problem, we turn to the technique of some auxiliary variables which reduces the higher-order derivatives in LABCs into a family of equations with lower-order derivatives. Furthermore, we extend the strategy to the stability analysis of two-dimensional problems by carefully dealing with the LABCs at corners. Numerical examples are given to demonstrate the effectiveness of our boundary conditions and validate the theoretical analysis.

Mathematics subject classification: 65M12, 65M06, 65M15.
Key words: Nonlinear Schrödinger equations, Energy estimates, Absorbing boundary conditions.

1. Introduction

In this paper we consider numerical solutions of nonlinear Schrödinger (NLS) equations for wave function $\psi(x,t)$, given by

$$i\partial_t\psi(x,t) = -\triangle\psi(x,t) + f(|\psi|,x,t)\psi(x,t), \quad x\in\mathbb{R}^d, \quad t>0, \tag{1.1}$$

with $d=1,2$. The nonlinear Schrödinger equation has been widely studied in fluid mechanics, nonlinear optics, atomic and molecular physics, for which, the nonlinear term $f(|\psi|,x,t)$ could

* Received April 7, 2014 / Revised version received August 27, 2016 / Accepted August 28, 2016 / Published online January 18, 2017 /

have different forms depending on practical applications [1–4]. For example, in fluid mechanics and optics, it usually appears as a cubic nonlinear Schrödinger (NLS) equation. In Bose-Einstein condensation [5], Eq. (1.1) is known as the Gross-Pitaevskii equation, where $f(|\psi|, x, t)$ is composed of a nonlinear potential and a harmonic trap potential. Recent interest of this class of Schrödinger equations also includes the time-dependent density functional theory [6] to investigate quantum many-body systems, for which the potential comes from the external field and the internal Coulomb interactions.

The wave function in the NLS equation is defined on an infinite domain. The design of suitable boundary conditions is essential for numerical computations on an interested truncated region, which is the main concern of the present work. Historically, the so-called absorbing (or artificial) boundary condition (ABC) has been widely studied for various types of linear PDEs for which the classical techniques such as the Fourier (Laplace) transform and the spherical harmonics expansion, are usually applicable (see recent reviews [7–9]). However, it is notoriously hard for treating nonlinear equations due to the lack of general tools, presenting an urgent requirement for people working on this field to develop new techniques and methodologies.

Some useful methods have been developed for designing ABCs of NLS equations. A class of these methods are by considering the physical material near the boundary as an artificial potential which absorbs outgoing waves. One technique is the perfectly matched layer (PML) [10–12], which had been applied to numerically solve NLS equations [13,14]. Another material-based method [15] is to add a negative imaginary potential as absorbing potential to the model equation, which has been often used in practical simulations.

Mathematically, there are two types of ABCs, say, nonlocal and local ABCs, which have been developed to treat both linear and nonlinear Schrödinger equations. For linear equations, the nonlocal condition, also called the transparent condition or DtN map, is exact and of integral form in time, and has been studied from different aspects such as analysis of discretization schemes [16], fast evaluation of the integral [17] and extension to multi-dimensions [18, 19]. The exact ABCs for NLS equations are only available for some very special cases, limited to one-dimensional version. A typical example is the integrable cubic NLS equation, which can be solved by the inverse scattering method [20]. Also, if the potential $f(|\psi|, x, t)$ is independent of the wave function and periodic outside the computational domain, efficient methods were developed in recent papers to construct exact ABCs (see, for example, [21, 22] and reference therein). For a general potential, it is well-known that one could not solve out an explicit representation of ABCs, since nonlinear interaction and the wave-potential interaction are too complex to clearly understand, and thus some simplification has to be applied. Under the assumption that the potential is slowly oscillatory and the density wave is of high-frequency propagation, a general approach [23–25] has been developed by extending the result of linear versions, and tested for many different potentials.

Local ABCs (LABCs) are usually constructed by extending the classical Engquist-Majda method [26] to factorize the Schrödinger operator in the linear equation, and to approximate the outgoing component by the Taylor or Padé expansions. The obtained differential equations can be naturally coupled with the nonlinear term by the time-splitting technique [28,29], which results in ABCs in discretization form. Recently, this approach was further developed into a general principle, called the unified approach [30–32], by recombining the subproblems of the time-splitting procedure into a continuous nonlinear differential equations as the effective LABCs. With these LABCs, the problem on unbounded domain is reformulated into a reduced problem on a bounded domain. To our knowledge, a rigorous mathematical analysis of the