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A SADDLE POINT NUMERICAL METHOD FOR HELMHOLTZ EQUATIONS*

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Abstract

In a previous work, the author and D.C. Dobson proposed a numerical method for solving the complex Helmholtz equation based on the minimization variational principles developed by Milton, Seppecher, and Bouchitté. This method results in a system of equations with a symmetric positive definite coefficient matrix, but at the same time requires solving simultaneously for the solution and its gradient. Herein is presented a method based on the saddle point variational principles of Milton, Seppecher, and Bouchitté, which produces symmetric positive definite systems of equations, but eliminates the necessity of solving for the gradient of the solution. The result is a method for a wide class of Helmholtz problems based completely on the Conjugate Gradient algorithm.

Mathematics subject classification: Primary 65N30, Secondary 35A15.
Key words: Helmholtz, Conjugate gradient, Saddle point, Finite element.

1. Introduction

The Helmholtz equation

$$\nabla \cdot L\nabla u = Mu,$$

is useful in modeling wave propagation in problems arising from many different physical situations. We will focus only on the homogeneous equation for simplicity and brevity, but the methods presented here can easily be extended to the non-homogeneous case. Suppose we wish to solve the Helmholtz equation in a domain $\Omega \subset \mathbb{R}^d$, and assume that L and M are complex-valued functions. A common source of numerical methods for solving this equation is the variational principle

$$\int_\Omega [-L\nabla u \cdot \nabla \bar{v} - Mu\bar{v}]\, dx = 0, \quad \forall\ v \in H_0^1(\Omega). \tag{1.1}$$

Since this is a stationary principle, the resulting system of equations is often indefinite, and indefinite systems are generally more difficult to solve than a system of equations having a positive definite coefficient matrix.

Because of the challenges in solving these indefinite systems, there has been much work devoted to solving the Helmholtz equation by replacing the indefinite systems with equivalent symmetric positive definite linear systems. Classical examples of such approaches are the CGNR and CGNE methods [1], based on solving normal equations associated with the original system. While such approaches produce positive definite systems, the normal equations are often poorly conditioned and preconditioning can be difficult. Another related approach

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is First Order System Least Squares (FOSLS) [2–4],which converts the second order equation into an equivalent system of first order equations and then solves a least squares problem for this system. The method presented here also produces positive definite systems of equations, but it does so without reformulation as a least squares problem.

When iterative methods are employed to solve a system of linear equations, it is usually necessary to precondition the original system in order to speed up convergence. A great deal of work has been dedicated to formulating effective preconditioning strategies for the linear systems resulting from discretizations of the Helmholtz equation [5]. One approach that has seen much success is the Shifted Laplacian preconditioner [6–8]. In this approach, the precoditioner for the system of equations corresponding to $\Delta u + k^2 u = 0$ is the matrix corresponding to the "shifted" equation $\Delta u + (\alpha + i\beta)u = 0$. If the imaginary shift β is large enough, multigrid methods are expected to be successful in solving the shifted problem, and if $\alpha \approx 1$, the shifted operator should be a good preconditioner for the original problem. While this approach is often effective, in [9] the authors point out the advantages in using a preconditioner that is symmetric positive definite. When the preconditioning matrix is not positive definite, the coefficient matrix of the preconditioned system is not symmetric with respect to any inner product, which limits the methods available for solving the resulting system. The solution suggested in [9] is to use an approximation of the absolute value of the original coefficient matrix as preconditioner. In the method proposed here, both the matrices and the suggested preconditioners are symmetric positive definite, and therefore a wide range of Krylov subspace methods is available. In particular, we shall demonstrate the results obtained with Conjugate Gradient, which has a short recurrence and is very simple to implement and parallelize.

As a background to this approach, we follow [10], where Milton, Seppecher, and Bouchitté developed variational principles that apply to the Helmholtz equation above, as well as the time-harmonic Maxwell equations and the equations of linear elasticity in lossy materials. To derive these variational principles, we first define the dual variable

$$v = iL\nabla u.$$

Then

$$\begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \nabla u \\ u \end{pmatrix} = \begin{pmatrix} L\nabla u \\ Mu \end{pmatrix} = \begin{pmatrix} -iv \\ -i\nabla \cdot v \end{pmatrix},$$

or equivalently,

$$\mathcal{G} = Z\mathcal{F},$$

where

$$\mathcal{F} = \begin{pmatrix} \nabla u \\ u \end{pmatrix}, \ \mathcal{G} = \begin{pmatrix} -iv \\ -i\nabla \cdot v \end{pmatrix}, \ Z = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}.$$

For a complex quantity z, we will write $z' = \mathrm{Re}(z)$ and $z'' = \mathrm{Im}(z)$. Taking real and imaginary parts, the constitutive relation becomes

$$\mathcal{G}' = Z'\mathcal{F}' - Z''\mathcal{F}'' \text{ and } \mathcal{G}'' = Z'\mathcal{F}'' + Z''\mathcal{F}',$$

which can be written in matrix form as

$$\begin{pmatrix} \mathcal{G}'' \\ \mathcal{G}' \end{pmatrix} = \begin{pmatrix} Z'' & Z' \\ Z' & -Z'' \end{pmatrix} \begin{pmatrix} \mathcal{F}' \\ \mathcal{F}'' \end{pmatrix}. \tag{1.2}$$