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ANALYSIS OF A NUMERICAL METHOD FOR RADIATIVE TRANSFER EQUATION BASED BIOLUMINESCENCE TOMOGRAPHY*

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Abstract

In the bioluminescence tomography (BLT) problem, one constructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface. The BLT problem is ill-posed and often the Tikhonov regularization is used to obtain stable approximate solutions. In conventional Tikhonov regularization, it is crucial to choose a proper regularization parameter to balance the accuracy and stability of approximate solutions. In this paper, a parameter-dependent coupled complex boundary method (CCBM) based Tikhonov regularization is applied to the BLT problem governed by the radiative transfer equation (RTE). By properly adjusting the parameter in the Robin boundary condition, we achieve one important property: the regularized solutions are uniformly stable with respect to the regularization parameter so that the regularization parameter can be chosen based solely on the consideration of the solution accuracy. The discrete-ordinate finite-element method is used to compute numerical solutions. Numerical results are provided to illustrate the performance of the proposed method.

Mathematics subject classification: 92C55, 65F22, 80M10.
Key words: Bioluminescence tomography, radiative transfer equation, Tikhonov regularization, coupled complex boundary method, convergence.

1. Introduction

Bioluminescence tomography (BLT) is a new molecular imaging modality and has shown its potential in monitoring non-invasively physiological and pathological processes *in vivo* at the cellular and molecular level. It is particularly attractive for *in vivo* applications because no external excitation source is needed and thus background noise is low while sensitivity is

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high ([38]). In the BLT problem, one reconstructs quantitatively the bioluminescence source distribution inside a small animal from optical signals detected on the animal's body surface.

A basic prerequisite for the BLT problem is the knowledge about the forward model describing the light propagation in the biological medium. Transmission of the bioluminescent photons through the biological medium is subject to both scattering and absorption, and is accurately described by the radiative transfer equation (RTE) ([2,5]). Since it is very challenging to solve the RTE accurately, diffusion approximation (DA) of the RTE is popularly used as the forward model. Plenty of references can be found in the literature on theoretical analysis and numerical simulations on the DA-based BLT problem, e.g. [11,17,21,25,32,37] and references therein for instance. However, as it is noted in [1], the DA is not always a good approximation of the RTE, especially when the scattering is relatively low. Higher order of approximate equations to the RTE such as $\mathrm{SP_N}$ and differential approximations etc. can be used to increase the approximation accuracy [23,30,39].

In this paper, we consider the more accurate RTE-based BLT problem. Let $X \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary ∂X and Ω be the unit sphere in $\mathbb{R}^3$. Denote by $\Gamma = \partial X \times \Omega$ the boundary of $X \times \Omega$, and by Γ_- and Γ_+ the incoming and outgoing parts of the boundary:

$$\Gamma_- := \Big\{(x,\omega) \in \Gamma \mid \omega \cdot \nu < 0\Big\}, \quad \Gamma_+ := \Big\{(x,\omega) \in \Gamma \mid \omega \cdot \nu > 0\Big\},$$

where $\nu := \nu(x)$ is the unit outward normal vector at $x \in \partial X$. With a normalized non-negative kernel function η:

$$\int_\Omega \eta(x, \omega \cdot \widehat{\omega}) d\sigma(\widehat{\omega}) = 1 \quad \forall\, x \in X,\, \omega \in \Omega,$$

define an integral operator S by

$$Su(x,\omega) = \int_\Omega \eta(x, \omega \cdot \widehat{\omega})\, u(x, \widehat{\omega})\, d\sigma(\widehat{\omega}).$$

In most applications, η is chosen to be independent of x. One well-known example is the 3D Henyey-Greestein phase function ([26])

$$\eta(t) = \frac{1-g^2}{4\pi(1+g^2-2gt)^{3/2}}, \quad t := \omega \cdot \widehat{\omega} \in [-1,1],$$

where $g \in (-1,1)$ is the anisotropy factor of the scattering medium: $g = 0$ for isotropic scattering, $g > 0$ for forward scattering, and $g < 0$ for backward scattering.

With an admissible set to be specified later, we consider the following inverse source problem:

Problem 1.1. *Given u_m on Γ_+, find a source function p from the admissible set so that the solution u of the boundary value problem (BVP)*

$$\begin{cases} \omega \cdot \nabla u(x,\omega) + \mu_t(x) u(x,\omega) = \mu_s(x)(Su)(x,\omega) + p(x)\chi_0(x), & (x,\omega) \in X \times \Omega, \\ u(x,\omega) = 0, & (x,\omega) \in \Gamma_- \end{cases} \tag{1.1}$$

matches the boundary measurement u_m for the density of outgoing photons:

$$u(x,\omega) = u_m(x,\omega), \quad (x,\omega) \in \Gamma_+.$$