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SYMPLECTIC SCHEMES FOR TELEGRAPH EQUATIONS*

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Abstract

A new numerical algorithm for telegraph equations with homogeneous boundary conditions is proposed. Due to the damping terms in telegraph equations, there is no royal conservation law according to Noether's theorem. The algorithm origins from the discovery of a transform applied to a telegraph equation, which transforms the telegraph equation to a Klein-Gordon equation. The Symplectic method is then brought in this algorithm to solve the Klein-Gordon equation, which is based on the fact that the Klein-Gordon equation with the homogeneous boundary condition is a perfect Hamiltonian system and the symplectic method works very well for Hamiltonian systems. The transformation itself and the inverse transformation theoretically bring no error to the numerical computation. Therefore the error only comes from the symplectic scheme chosen. The telegraph equation is finally explicitly computed when an explicit symplectic scheme is utilized. A relatively long time result can be expected due to the application of the symplectic method. Meanwhile, we present order analysis for both one-dimensional and multi-dimensional cases in the paper. The efficiency of this approach is demonstrated with numerical examples.

Mathematics subject classification: 65M06, 65P10.
Key words: Telegraph equation, Klein-Gordon equation, Symplectic method, Explicit method.

1. Introduction

Telegraph equation, also known as transmission line equation, describes the rule within transmission lines. Transmission lines, such as parallel line (ladder line), coaxial cable, stripline, and microstrip, have broad applications in signal transfer, pulse generation, stub filters, and so on. There are generally two means of solving telegraph equations. One is seeking for exact solutions, for instance, by Fourier transform. The exact solution, however, is only available for simple cases. The other is seeking for numerical solutions. Based on the way of solving the equation numerically, we can divide them into two groups, analytical method and numerical method. An analytical method solves the equation by iteration of waveform. The waveform goes close to the iterative solution. The well known analytical methods are Exp-function method [11], Adomian decomposition method (ADM) [14], variational iteration method (VIM) [16] and homotopy perturbation method (HPM) [13]. A numerical method, however, solves the equation on space and time grids. Numerical methods known for telegraph equation include differential quadrature method (DQM) [8], alternating direction implicit (ADI) scheme [4], and weighted essentially non-oscillatory (WENO) method [1, 15].

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In this paper, we are interested in numerical methods for the following telegraph equation

$$\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} + k\dfrac{\partial w(x,t)}{\partial t} = a^2 \dfrac{\partial^2 w(x,t)}{\partial x^2} + bw(x,t), \\ w(x,0) = g_1(x), \qquad 0 \le x \le 1, \\ w_t(x,0) = g_2(x), \qquad 0 \le x \le 1, \\ w(0,t) = 0, \qquad 0 \le t \le T, \\ w(1,t) = 0, \qquad 0 \le t \le T, \end{cases} \tag{1.1}$$

where $k > 0$, $a > 0$ and $b < 0$ are constants. $g_1(x)$ and $g_2(x)$ are smooth functions, with compatibility condition $g_1(0) = 0$. $w(x,t) \in \mathbb{R}$ is the sought function. This equation is derived from the transmission line problem. An equivalent circuit of the transmission line problem is shown in Fig. 1.1, where and for (1.1), we have

$$a^2 = \frac{1}{L_1 C_1}, \quad k = \left(\frac{R_2^{-1}}{C_1} + \frac{R_1}{L_1}\right), \quad b = -\frac{R_1 R_2^{-1}}{L_1 C_1}.$$

As far as we know, with a transformation, the telegraph equation (1.1) can be turned into a system with conservation law. Besides, with the conservation law, we can solve it by symplectic methods. The advantages of this choice are better long term behavior and the potential explicit implementation. The transformation is covered in section 2.1.

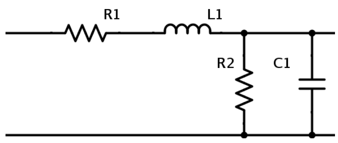

Fig. 1.1. A schematic representation of the elementary components of a transmission line.

The finite difference method of Eq. (1.1) can be done in two ways. One is taking a full discretization of the equation, that is, taking an equispaced grid $0 = x_0 < x_1 < \cdots < x_N = 1,\ 0 = t_0 < t_1 < \cdots < t_n = T$. The full discretization is,

$$\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} + k\frac{w_i^{j+1} - w_i^j}{\Delta t} = a^2 \frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{\Delta x^2} + bw_i^{j+1}, \tag{1.2}$$

where $1 \le i \le N-1$ and $1 \le j \le n-1$ in w_i^j stand for space and time grid point respectively, i.e., $w(x_i, t_j) = w_i^j$. The other is taking a semi-discretization. Taking an equispaced grid in space $0 = x_0 < x_1 < \cdots < x_N = 1$, we get a group of ODEs,

$$\frac{d^2 W_i}{dt^2} + k\frac{dW_i}{dt} = a^2 \frac{W_{i+1} - 2W_i + W_{i-1}}{\Delta x^2} + bW_i,$$

where $1 \le i \le N-1$, and $W_i = W(x_i, t)$. Here, space discretization is taken as of order 2. Setting $W = (W_1, W_2, \cdots, W_{N-1})^T$, we get

$$\frac{d^2 W}{dt^2} + k\frac{dW}{dt} = -\frac{a^2}{\Delta x^2} SW + bW, \tag{1.3}$$