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CONVERGENCE RATES OF MOVING MESH RANNACHER METHODS FOR PDES OF ASIAN OPTIONS PRICING*

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Abstract

This paper studies the convergence rates of a moving mesh implicit finite difference method with interpolation for partial differential equations (PDEs) with moving boundary arising in Asian option pricing. The moving mesh scheme is based on Rannacher time-stepping approach whose idea is running the implicit Euler schemes in the initial few steps and continuing with Crank-Nicolson schemes. With graded meshes for time direction and moving meshes for space direction, the fully discretized scheme is constructed using quadratic interpolation between two consecutive time level for the PDEs with moving boundary. The second-order convergence rates in both time and space are proved and numerical examples are carried out to confirm the theoretical results.

Mathematics subject classification: 65M06, 65M12, 91G20, 91G60, 91G80.
Key words: Asian option pricing, Moving mesh methods, Crank-Nicolson schemes, Rannacher time-stepping schemes, Convergence analysis.

1. Introduction

In this paper we study the valuation of continuous-time arithmetic Asian options with fixed strikes. Since the valuation of such an option does not have analytical solutions, numerical methods are the necessary tools to solve the problem. Numerical PDE method is one of the most popular numerical methods. The problems are described as follows. Let the stock price $S = S(t)$ follow a geometric Brownian motion in the risk-neutral sense

$$dS(t) = rS(t)dt + \sigma S(t)dW(t),$$

where r denotes the risk free interest rate, σ the volatility, and $dW(t)$ the standard Brownian motion. Denote by

$$I(t) = \int_0^t S(\tau)\, d\tau.$$

Then the value of the continuous arithmetic average Asian options with payoff $\max(I(T)/T-K, 0)$, where K is the fixed strike and T is the expiry time of the option, is given by

$$C(S(t), I(t), t) = e^{-r(T-t)} E_t \left[\max\left(I(T)/T - K, 0\right)\right],$$

where E_t denotes the conditional expectation at t. As well-known (see e.g., Ingersoll [10]), the value of the Asian option is formulated to satisfy the following PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + S\frac{\partial C}{\partial I} - rC = 0, \tag{1.1}$$

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where $C = C(S, I, t)$ with S and I being dummy variables. The terminal condition at expiry time T is given by

$$C(S, I, T) = \max\left(I/T - K, 0\right). \tag{1.2}$$

PDE (1.1) is a two-dimensional problem and there is no diffusion in the I direction. These facts cause many difficulties in the numerical solutions with standard finite difference methods.

Using transformation of variables

$$\xi = \frac{K - I/T}{S}, \quad w(\xi, t) = \frac{C}{S},$$

Rogers and Shi [20] reduce the two-dimensional PDE (1.1) into the following one dimensional PDE:

$$-\frac{\partial w}{\partial t} - \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 w}{\partial \xi^2} + \left(\frac{1}{T} + r\xi\right)\frac{\partial w}{\partial \xi} = 0, \tag{1.3}$$

with terminal condition $w(\xi, T) = \max\{-\xi, 0\}$. Although this PDE (1.3) is one-dimensional, the standard finite difference method is still difficult to use since when ξ is close to 0 and for short expiry time T, the convection term dominates the diffusion term. Zvan et al. [26] construct a total variation diminishing (TVD) scheme, whose idea originates from the field of computational fluid dynamics, for solving the one dimensional PDE (1.3) and the two dimensional PDE (1.1). But there is no theoretical analysis of the convergence rates for the TVD scheme in the paper by Zvan et al. [26].

Dubois and Lelièvre [6] introduce another transformation of variables

$$x = \frac{t}{T} + \frac{K - I/T}{S}, \quad u(x, t) = \frac{C}{S}, \tag{1.4}$$

to (1.1) and give a new form of one-dimensional PDE:

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2\left(x - \frac{t}{T}\right)^2 \frac{\partial^2 u}{\partial x^2} + r\left(x - \frac{t}{T}\right)\frac{\partial u}{\partial x} = 0, \quad \text{for } x \in (-\infty, +\infty), \tag{1.5}$$

with terminal condition $u(x, T) = \max\left(1 - x, 0\right)$. This PDE has been also obtained by Večeř [23] using stochastic analysis tools. This PDE (1.5) is free of convection-domination. Dubois and Lelièvre [6] develop a Crank-Nicolson scheme to solve (1.5), which is based on fixed meshes in the space direction and uniform meshes in the time direction. The convergence rates are not analyzed in their paper.

Due to the degeneracy of the PDE (1.1) in the I direction, it can be verified that in the region $I \geq KT$, for all $t \leq T$,

$$C(S, I, t) = \frac{S}{rT}\left(1 - e^{-r(T-t)}\right) + \left(\frac{I}{T} - K\right)e^{-r(T-t)}, \tag{1.6}$$

satisfies the PDE (1.1) (see e.g., Geman and Yor [7], Dubois and Lelièvre [6]). Using the transformation of variables (1.4), formula (1.6) in the region $I \geq KT$ is re-written as

$$u(x, t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) - \left(x - \frac{t}{T}\right)e^{-r(T-t)}, \quad \text{for } x \in \left(-\infty, \frac{t}{T}\right].$$

Therefore, we may simply solve the (1.5) in the region $(\frac{t}{T}, +\infty)$ instead of the whole region $(-\infty, +\infty)$. Furthermore for convenience, we may transform the terminal-value problem into