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LOCAL SUPERCONVERGENCE OF CONTINUOUS GALERKIN SOLUTIONS FOR DELAY DIFFERENTIAL EQUATIONS OF PANTOGRAPH TYPE*

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Abstract

This paper is concerned with the superconvergent points of the continuous Galerkin solutions for delay differential equations of pantograph type. We prove the local nodal superconvergence of continuous Galerkin solutions under uniform meshes and locate all the superconvergent points based on the supercloseness between the continuous Galerkin solution U and the interpolation $\Pi_h u$ of the exact solution u. The theoretical results are illustrated by numerical examples.

Mathematics subject classification: 65L60, 65N70.
Key words: Pantograph delay differential equations, Uniform mesh, Continuous Galerkin methods, Supercloseness, Superconvergence.

1. Introduction

The present paper is motivated by recent work of Huang et al. [10] on the question of optimal local superconvergence in discontinuous Galerkin methods for delay differential equation (DDE) with proportional delay, also called the pantograph equation

$$
\begin{aligned}
&u'(t) = a(t)u(t) + b(t)u(qt) + f(t), \qquad t \in J = [0,T], \qquad 0 < q < 1, \\
&u(0) = u_0.
\end{aligned} \tag{1.1}
$$

It is well known that DDEs with proportional delay have been widely investigated analytically and numerically. See, for example, Iserles [11] presented an introduction and pointed out some challenges in numerical analysis of such DDEs. Liu [16] used a perturbation method and a simple numerical discretization to get the correct asymptotic behaviour of the exact solution for DDEs of pantograph type. Some other numerical methods, such as linear multistep method [3], Runge-Kutta method [1, 12], collocation method [19], were also considered extensively. The monographs by Bellen and Zennaro [2] and Brunner [4] also conveyed good pictures of various numerical methods and relevant analysis for DDEs of pantograph type.

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Finite element (FE) methods are efficient numerical methods in solving various partial differential equations and integral equations. Superconvergence of the FE methods is a phenomenon that the convergence rate of FE approximations exceeds what is globally at some special points. There have been many studies concerning with superconvergent points of FE methods in recent years. See, for example, monographs [6, 14, 15]. Zhang etc. [21, 22] analyzed local natural superconvergent points of FE methods in 3D and natural superconvergent points of equilateral triangular FEs.

FE methods were also applied in solving ordinary and delay differential equations. See, for example, discontinuous Galerkin (DG) methods and continuous Galerkin (CG) methods for ordinary differential equations [7, 9, 17, 18], for DDEs with constant delay [8, 13], and for DDEs with proportional delay [5,20]. Among them, Pan and Chen [17] got the nodal superconvergence of the CG solutions of ODEs; the superconvergent points of CG and DG solutions are Lobatto [8] and Radau II points [13] respectively for DDEs of constant delay; for DDEs of pantograph type, the superconvergent points of DG solutions are Radau II points [10]. Thus, there arises the question as to what these local superconvergence orders are and where the superconvergent points are located for CG solutions of DDEs of pantograph type.

The main purpose of our current work is to study the local superconvergence of CG solutions for DDEs of pantograph type. We will show that the superconvergent points are the Lobatto points and the optimal order (at the mesh and Lobatto points) of the CG solution of piecewise polynomial of degree m ($m \geq 2$) equals to $m{+}2$ with uniform stepsize.The optimal order $O(h^{2m})$ at the mesh points for DDEs with proportional delay is obtained only by some quasi-geometric mesh. We will leave it to another paper.

This paper is arranged as follows. Section 2 introduces the CG method and reviews some relevant knowledge. In Section 3, we present the main results: superconvergence of CG solutions at mesh points, supercloseness between the CG solution and a suitable interpolant of the exact solution, and the location of the superconvergent points. In Section 4, we provide some numerical examples to support our theoretical results. Finally, in Section 5, we make concluding remarks and list some possible future research work.

2. Continuous Galerkin Method for Delay Differential Equations with Proportional Delay

Let $J_N : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition for the given interval $J = [0, T]$ and set

$$I_n = (t_{n-1}, t_n), \quad h_n = t_n - t_{n-1} \quad (1 \leq n \leq N), \quad h = \max_{1 \leq n \leq N} h_n.$$

We choose the following CG finite element space as

$$S_m^{(0)}(J_N) = \{v \in C(J) : v|_{I_n} \in P_m, 1 \leq n \leq N\}.$$

Where P_m denotes the set of polynomials of degree not exceeding m with $m \geq 1$.
Then m-degree CG solution $U \in S_m^{(0)}(J_N)$ can be defined by

$$\sum_{n=1}^{N} \int_{I_n} U'(t)v(t)dt = \sum_{n=1}^{N} \int_{I_n} [a(t)U(t) + b(t)U(qt) + f(t)]v(t)dt, \quad \forall v \in S_{m-1}^{(0)}(J_N). \tag{2.1}$$

Here, $v \in S_{m-1}^{(0)}(J_N)$ is because of the continuity of U. That is,

$$U(t_{n-1}) = \lim_{t \to t_{n-1}^-} U(t) = \lim_{t \to t_{n-1}^+} U(t),$$