Journal of Computational Mathematics
Vol.33, No.3, 2015, 307–322.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1502-m4509

A MODIFIED WEAK GALERKIN FINITE ELEMENT METHOD FOR SOBOLEV EQUATION*

Fuzheng Gao
School of Mathematics, Shandong University, Jinan, Shandong 250100, China
School of Materials Science and Engineering, Shandong University, Jinan, Shandong, China
Email: fzgao@sdu.edu.cn
Xiaoshen Wang
Department of Mathematics, University of Arkansas at Little Rock, 2801 S. University Avenue, Little Rock, AR 72204, USA.
E-mail: xxwang@ualr.edu

Abstract

For Sobolev equation, we present a new numerical scheme based on a modified weak Galerkin finite element method, in which differential operators are approximated by weak forms through the usual integration by parts. In particular, the numerical method allows the use of discontinuous finite element functions and arbitrary shape of element. Optimal order error estimates in discrete H^1 and L^2 norms are established for the corresponding modified weak Galerkin finite element solutions. Finally, some numerical results are given to verify theoretical results.

Mathematics subject classification: 65M15, 65M60.
Key words: Galerkin FEMs, Sobolev equation, Discrete weak gradient, Modified weak Galerkin, Error estimate

1. Introduction

Sobolev equation is a classical partial differential equation, which includes a third order mixed derivative with respect to time and space. It is used to describe wave motion in media with nonlinear wave steepening and balancing with dispersion and diffusion, which are important not only in hydrodynamics but also in many other disciplines of engineering and science.

We consider the following Sobolev equation

$$\begin{aligned} &U_t - \nabla\cdot(\mu\nabla U_t + \varepsilon\nabla U) = f(x,t), && x\in\Omega, \quad t\in(0,T], \\ &U = \mathbb{u}(x), && x\in\Omega, \quad t=0, \end{aligned} \tag{1.1}$$

with homogenous Dirichlet boundary condition, where $U(x,t)$ is the unknown solution in Ω, which is a bounded domain in R^2 with boundary $\partial\Omega$. $\mathbb{u}(x)$, $f(x,t)$ are given sufficiently smooth scalar functions. $\varepsilon \geq 0$ and $\mu \geq \mu_0 > 0$, for fixed constant μ_0, are diffusion constant coefficient and dispersion constant coefficient.

A large number of works on finite element methods (FEMs) have been done for Sobolev equations, for example, standard FEMs [1,6,7,15,17], mixed FEMs [12,16], and Petrov-Galerkin methods [2, 5]. Recently, we proposed semi-discrete LDG scheme, as well as fully-discrete

* Received April 10, 2014 / Revised version received September 19, 2014 / Accepted February 4, 2015 / Published online May 15, 2015 /

LDG scheme with the second and/or third order explicitly total variation diminishing Runge-Kutta (TVDRK2) time-marching in [8] and [22] for the model problem (1.1), respectively. We obtained quasi-optimal and optimal a priori error estimates for both considered schemes. We also proposed for Sobolev equation with convection term split least-squares characteristic mixed finite element method in which primal variable and flux introduced can be solved, separately [9]. Theoretical result shows that the method yields approximate solutions with optimal order accuracy in $L^2(\Omega)$ norm for primal variable and in $H(div;\Omega)$ norm for flux, respectively.

Our purpose here is to propose and analyze semi-discrete and fully-discrete finite element procedures for Sobolev equation (1.1) using a modified weak Galerkin finite element method (MWG-FEM). The weak Galerkin finite element method (WG-FEM) was introduced by Wang and Ye [19], and was developed in [10, 14, 20]. The key idea lies in the approximation of differential operators by weak forms for discontinuous finite element functions defined on a partition of the domain. Comparing with traditional Galerkin FEMs, the WG-FEMs allow the use of a new class of discontinuous finite element functions in the algorithm design, similar to discontinuous Galerkin finite element method (DG-FEM) [4], but remove the need for picking some "large enough" parameters. However, in the WG-FEM [10, 19], each function has two components, interior and boundary, which adds the degree of freedom. We introduce a new definition of weak gradient in [11], which does not need artificial boundary component of a function and adopt a new stabilization term which are different from those of [10, 19].

The remainder of this paper is organized as follows. In section 2, we give the detailed implementation and the stability theorems for the semi-discrete and fully-discrete MWG method with the backward Euler time-marching and some lemmas. In section 3, we present error equation and some main theorems for the MWG discretization. Finally, some numerical experiments are given in section 4.

2. The MWG-FEM and Some Lemmas

In this section, we present semi- and fully-discrete MWG schemes for the model problem (1.1), where the time is updated by the backward Euler time-marching.

The variational weak form of (1.1) is: Find $U = U(x,t) \in H_0^1(\Omega)\,(0 \leqslant t \leqslant T)$, such that

$$(U_t, v) + \mu A(U_t, v) + \varepsilon A(U, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \quad t > 0, \tag{2.1a}$$

$$U(x, 0) = \mathbb{u}(x), \qquad x \in \Omega, \tag{2.1b}$$

where $(\cdot,\cdot)$ denotes inner product of $L^2(\Omega)$ and the bilinear form $A(\cdot,\cdot)$ is defined by

$$A(U, v) := \int_\Omega \nabla U \cdot \nabla v \mathrm{d}x. \tag{2.2}$$

2.1. MWG finite element space

First, we use the standard definition for the Sobolev space $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$ and seminorms $|\cdot|_{s,D}$ for any $s \geq 0$. We shall drop the subscript D when $D = \Omega$ and s as $s = 0$ in the norm and inner product notation.

Let $\mathcal{T}_h$ be a quasi-uniform partition of the domain Ω consisting of polygons satisfying a set of conditions specified in [13, 19]. Please note that T_h does not have to be conforming. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges. For every element $K \in \mathcal{T}_h$, we denote by h_K its diameter and mesh size $h = max_{K \in \mathcal{T}_h} h_K$ for $\mathcal{T}_h$.