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FINITE ELEMENT APPROXIMATIONS OF SYMMETRIC TENSORS ON SIMPLICIAL GRIDS IN $\mathbb{R}^n$: THE HIGHER ORDER CASE*

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Abstract

The design of mixed finite element methods in linear elasticity with symmetric stress approximations has been a longstanding open problem until Arnold and Winther designed the first family of mixed finite elements where the discrete stress space is the space of $H(\text{div}, \Omega; \mathbb{S})$—$P_{k+1}$ tensors whose divergence is a P_{k-1} polynomial on each triangle for $k \geq 2$. Such a two dimensional family was extended, by Arnold, Awanou and Winther, to a three dimensional family of mixed elements where the discrete stress space is the space of $H(\text{div}, \Omega; \mathbb{S})$—$P_{k+2}$ tensors, whose divergence is a P_{k-1} polynomial on each tetrahedron for $k \geq 2$. In this paper, we are able to construct, in a unified fashion, mixed finite element methods with symmetric stress approximations on an arbitrary simplex in $\mathbb{R}^n$ for any space dimension. On the contrary, the discrete stress space here is the space of $H(\text{div}, \Omega; \mathbb{S})$—$P_k$ tensors, and the discrete displacement space here is the space of $L^2(\Omega; \mathbb{R}^n)$—$P_{k-1}$ vectors for $k \geq n+1$. These finite element spaces are defined with respect to an arbitrary simplicial triangulation of the domain, and can be regarded as extensions to any dimension of those in two and three dimensions by Hu and Zhang.

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Key words: Mixed finite element, Symmetric finite element, First order system, Simplicial grid, Inf-sup condition.

1. Introduction

In the classical Hellinger-Reissner mixed formulation of the elasticity equations, the stress is sought in $H(\text{div}, \Omega; \mathbb{S})$ and the displacement in $L^2(\Omega; \mathbb{R}^2)$ for two dimensions and in $L^2(\Omega; \mathbb{R}^3)$ for three dimensions. The constructions of stable mixed finite elements using polynomial shape functions are a long-standing and challenging problem, see [4, 6]. To overcome this difficulty, earliest works adopted composite element techniques or weakly symmetric methods, cf. [3, 7, 8, 31, 33, 35–37]. In [10], Arnold and Winther designed the first family of mixed finite element methods in 2D, based on polynomial shape function spaces. From then on, various stable mixed elements have been constructed, see e.g., [2, 5, 6, 9–13, 18, 20, 22–24, 26–28, 32, 39, 40].

For first order systems with symmetric tensors in any space dimension, as the displacement u is in $L^2(\Omega; \mathbb{R}^n)$, a natural discretization is the piecewise P_{k-1} polynomial without interelement continuity. Even for two and three dimensional cases, it is a surprisingly hard problem if the stress tensor can be discretized by an appropriate P_k finite element subspace of $H(\text{div}, \Omega; \mathbb{S})$. In fact, in [10], see also [4], Arnold and Winther designed the first family of mixed finite elements

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where the discrete stress space is the space of $H(\mathrm{div},\Omega;\mathbb{S})$—$P_{k+1}$ tensors whose divergence is a P_{k-1} polynomial on each triangle for $k \geq 2$. Such a two dimensional family was extended to a three dimensional family of mixed elements where the discrete stress space is the space of $H(\mathrm{div},\Omega;\mathbb{S})$—$P_{k+2}$ tensors for $k \geq 2$, while the lowest order element with $k = 2$ was first proposed in [2]. In very recent papers [29] and [30], Hu and Zhang attacked this open problem by constructing a suitable $H(\mathrm{div},\Omega;\mathbb{S})$—$P_k$, instead of above P_{k+1} (2D, $k \geq 3$) or P_{k+2} (3D $k \geq 4$), finite element space for the stress discretization. The analysis there is based on a new idea for analyzing the discrete inf–sup condition. More precisely, they first decomposed the discontinuous displacement space into a subspace containing lower order polynomials and its orthogonal complement space. Second they found that the discrete stress space contains the full C^0-P_k space and some so-called $H(\mathrm{div})$ bubble function space on each triangle (2D) or tetrahedron (3D). Third they proved that the full C^0-P_k space can control the subspace containing lower order polynomials while the $H(\mathrm{div})$ bubble function space is able to deal with that orthogonal complement space. We refer interested readers to Hu [25] for similar mixed elements on rectangular and cuboid meshes.

The purpose of this paper is to generalize, in a unified fashion, the elements in [29] and [30] to any dimension. The analysis here is based on three key ingredients. First, based on the tangent vectors of a simplex, we construct $\frac{1}{2}n(n+1)$ symmetric matrices of rank one and prove that they are linearly independent and consequently form a basis of the space $\mathbb{S}$. Second, by using these matrices of rank one, we define a $H(\mathrm{div})$ bubble function space consisting of polynomials of degree $\leq k$ on each element and prove that it is indeed the full $H(\mathrm{div})$ bubble function space of order k. Third, we show that the divergence space of the $H(\mathrm{div})$ bubble function space is equal to the orthogonal complement space of the rigid motion space with respect to the discrete displacement on each element. We stress that such a result holds for any $k \geq 1$.

The rest of the paper is organized as follows. In the next section, we define finite element spaces of symmetric tensors in any space dimension, present a crucial structure of them, and a set of local degrees of freedom of shape function spaces on each element. We also prove that the divergence of the $H(\mathrm{div})$ bubble function space is equal to the orthogonal complement space of the rigid motion space with respect to the discrete displacement on each element. In Section 3, we apply these spaces to first order systems with symmetric tensors and prove the well–posedness of the discrete problem. The paper ends with Section 4, which gives some conclusion.

2. Finite Elements for Symmetric Tensors

We consider mixed finite element methods of first order systems with symmetric tensors: Find $(\sigma,u) \in \Sigma \times V := H(\mathrm{div},\Omega;\mathbb{S}) \times L^2(\Omega;\mathbb{R}^n)$, such that

$$\begin{cases} (A\sigma,\tau) + (\mathrm{div}\tau,u) = 0 & \text{for all } \ \tau \in \Sigma, \\ (\mathrm{div}\sigma,v) = (f,v) & \text{for all } \ v \in V. \end{cases} \tag{2.1}$$

Here the symmetric tensor space for the stress Σ is defined by

$$H(\mathrm{div},\Omega;\mathbb{S}) := \left\{ \tau = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1n} \\ \vdots & \vdots & \vdots \\ \tau_{n1} & \cdots & \tau_{nn} \end{pmatrix} \in H(\mathrm{div},\Omega;\mathbb{R}^{n\times n}),\ \tau^T = \tau \right\}, \tag{2.2}$$