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OPTIMAL AND PRESSURE-INDEPENDENT L^2 VELOCITY ERROR ESTIMATES FOR A MODIFIED CROUZEIX-RAVIART STOKES ELEMENT WITH BDM RECONSTRUCTIONS*

C. Brennecke
Eidgenössische Technische Hochschule Zürich, Departement Mathematik, Rämistr. 101, 8092 Zürich, Switzerland
Email: cbrenne@student.ethz.ch
A. Linke and C. Merdon
Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany
Email: Alexander.Linke@wias-berlin.de, Christian.Merdon@wias-berlin.de
J. Schöberl
TU Wien, Institut für Analysis und Scientific Computing, Wiedner Hauptstr. 8-10/101,1040 Wien, Austria
Email: joachim.schoeberl@tuwien.ac.at

Abstract

Nearly all inf-sup stable mixed finite elements for the incompressible Stokes equations relax the divergence constraint. The price to pay is that a priori estimates for the velocity error become pressure-dependent, while *divergence-free* mixed finite elements deliver *pressure-independent* estimates. A recently introduced new variational crime using lowest-order Raviart-Thomas velocity reconstructions delivers a much more robust modified Crouzeix-Raviart element, obeying an optimal *pressure-independent* discrete H^1 velocity estimate. Refining this approach, a more sophisticated variational crime employing the lowest-order BDM element is proposed, which also allows proving an optimal pressure-independent L^2 *velocity error*. Numerical examples confirm the analysis and demonstrate the improved robustness in the Navier-Stokes case.

Mathematics subject classification: 65N30, 65N15, 76D07.
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1. Introduction

The success of classical mixed finite elements for the incompressible Navier-Stokes equations relies heavily on the relaxation of the divergence constraint (mass conservation), enabling the construction of large classes of inf-sup stable finite element pairs for the approximation of velocity and pressure [2]. Unfortunately, this relaxation is not for free. In the simplest case, the incompressible Stokes equations

$$-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f},\quad \nabla\cdot\mathbf{u}=0, \tag{1.1}$$

the classical a priori error estimate for the velocity error [2,10] reads (for homogeneous Dirichlet boundary conditions)

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\le C_1\inf_{\mathbf{w}\in X_h}\|\mathbf{u}-\mathbf{w}_h\|_{1,h}+\frac{C_2}{\nu}\inf_{q_h\in Q_h}\|p-q_h\|_0. \tag{1.2}$$

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Divergence-free mixed finite element methods like the Scott-Vogelius finite element method deliver the *pressure-independent* and therefore significantly more robust estimate [3, 10]

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \leq C_3 \inf_{\mathbf{w}_h \in X_h} \|\mathbf{u} - \mathbf{w}_h\|_{1,h}. \tag{1.3}$$

In many physical situations, where the pressure is comparably small w.r.t. the velocity or approximable by low-order polynomials, the appearance of the pressure in the estimate (1.2) is indeed negligible. In general situations, however, mixed methods suffer from so-called *poor mass conservation*, which just means large velocity errors due to the pressure-dependent error term $C_2/\nu \inf_{q_h \in Q_h} \|p - q_h\|_0$. Note, that for conforming mixed finite element methods like the Taylor–Hood finite element method *poor mass conservation* is accompanied by large divergence errors [9]. The easiest example, where mixed methods reveal their lack of robustness, is the *no-flow example* [6,8,12], where one prescribes $\mathbf{f} = \nabla \phi$ as the forcing in (1.1). For homogeneous Dirichlet boundary conditions, $(\mathbf{u}, p) = (\mathbf{0}, \phi)$ uniquely solves (1.1). Obviously, in this example the pressure $p = \phi$ is *not small* compared to the velocity $\mathbf{u} = \mathbf{0}$. According to (1.3), divergence-free methods, deliver indeed a discrete velocity $\mathbf{u}_h = \mathbf{0}$, while mixed methods with a relaxed divergence constraint have a velocity error, which can be arbitrarily large, only dependent on ϕ, ν and the applied mixed method. Since the continuous velocity $\mathbf{u} = \mathbf{0}$ lies in the approximation space of the discrete method, mixed methods indeed suffer from a *stability problem.*

The traditional notion *poor mass conservation* is derived from conforming mixed methods like the Taylor-Hood element, where it is accompanied by large divergence errors. This numerical instability has been observed by several authors in the past. In [6] the *no-flow* example was investigated for the first time, seemingly. In [8] a numerical Helmholtz decomposition of the forcing $\mathbf{f}$ in (1.1) was applied, in order to get around with the irrotational part of $\mathbf{f}$. The standard approach for stabilizing *poor mass conservation* is the so-called grad-div stabilization [7, 15, 16], which penalizes divergence errors in an L^2 sense. Unfortunately, it can be shown that even in the simplest case of the incompressible Stokes equations with an optimal choice of the stabilization parameter, the approach is not completely robust w.r.t. small kinematic viscosities ν [11]. More in the spirit of [8], recently in [14] a new approach has been proposed, in order to avoid *poor mass conservation* completely. The approach is based on the observation that the proper source of the numerical instability is a *poor momentum balance*, where irrotational and divergence-free forces interact in a non-physical manner. Due to their L^2-orthogonality, divergence-free and irrotational forces are balanced separately in the continuous equations. But due to the relaxation of the divergence constraint in mixed methods, this separation fails in mixed methods, in general.

In [14] it is shown how to reestablish L^2-orthogonality between discretely divergence-free and irrotational vector fields modifying the nonconforming Crouzeix-Raviart element [5] by a variational crime. Here, a velocity reconstruction operator maps *discretely divergence-free test functions* onto *divergence-free lowest-order Raviart-Thomas functions* [17] in the right hand side of the incompressible Stokes equations. Replacing the test functions by these reconstructions introduces an additional consistency error, but improves the robustness of the Crouzeix-Raviart element, since one can prove the *pressure-independent*, a priori discrete H^1 velocity error estimate (1.3) as done in [14]. Unfortunately, in [14] the author did not succeed in proving also an optimal a priori L^2 error estimate for the velocity, although numerical experiments show that such an estimate probably holds. The proof of an optimal L^2 velocity error is non-trivial, since divergence-free lowest-order Raviart-Thomas elements are piecewise constant, only, and the variational crime committed is similar to the replacement of an exact integration by a numerical