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A CHEBYSHEV-GAUSS SPECTRAL COLLOCATION METHOD FOR ODRINARY DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we introduce an efficient Chebyshev-Gauss spectral collocation method for initial value problems of ordinary differential equations. We first propose a single interval method and analyze its convergence. We then develop a multi-interval method. The suggested algorithms enjoy spectral accuracy and can be implemented in stable and efficient manners. Some numerical comparisons with some popular methods are given to demonstrate the effectiveness of this approach.

Mathematics subject classification: 65L05, 65L60, 41A10.
Key words: Initial value problems of ordinary differential equations, Chebyshev-Gauss spectral collocation method, Spectral accuracy.

1. Introduction

A considerable amount of literature has been devoted to numerical solutions of ordinary differential equations (ODEs), see, e.g., [6,8,23–25,29,39]. For Hamiltonian systems, the interested reader may refer to [12,13,22,36]. Among the existing methods, numerical schemes based on Taylor's expansions or quadrature formulas have been frequently used, see, e.g., [7,8,23,24,29], which can be systematically designed and often provide accurate approximations.

In the past few decades, spectral method has become increasingly popular and been widely used in spatial discretization of PDEs owing to its spectral accuracy (i.e., the smoother the exact solutions become, the smaller the numerical errors will be), see, e.g., [4,5,9,14–16,37,38]. Moreover, some spectral methods for time discretization of PDEs have been developed rapidly, see, e.g., [2,3,11,27,30,31,40–43]. Recently, Guo et al. [18,19,45] developed several Legendre-Gauss-type spectral collocation methods for ODEs. Meanwhile, Guo et al. [20,21] designed Laguerre-Gauss-type spectral collocation methods for ODEs. Kanyamee and Zhang [28] also conducted a systematic comparison of a Legendre (Chebyshev)-Gauss-Lobatto spectral collocation method with some symplectic methods in solving Hamiltonian dynamical systems. For the $hp-$version of the continuous Galerkin FEM, we refer the reader to [46] and the references therein for other earlier works.

In this paper, we propose a Chebyshev-Gauss spectral collocation method for ODEs:

$$\begin{cases} \dfrac{d}{dt}U(t) = f(U(t),t), & 0 < t \leq T, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

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where f is a given function, and U_0 is the initial data.

We start with a single interval scheme. We approximate the solution by a finite Chebyshev series, and collocate the numerical scheme at Chebyshev-Gauss points to determine the coefficients. We introduce two algorithms. Numerical results show that the suggested algorithms can be implemented efficiently. Particularly, the algorithms are numerically stable and possess spectral accuracy. It is noted that Vigo-Aguiar and Ramos [44] also constructed a special family of Runge-Kutta collocation algorithms based on Chebyshev-Gauss-Lobatto points, with A-stability and stiffly accurate characteristics. The interested reader may also refer to [34, 35] for additional information.

For a more effective implementation, we also suggest a multi-interval scheme due to the following considerations:

- The resultant system for the expansion coefficients can be solved more efficiently for a modest number of unknowns. For large T, it is desirable to partition the solution interval $(0, T)$ and solve the subsystems successively. Hence, the scheme can be implemented efficiently and economically.
- For ensuring the convergence of the numerical scheme, the length of T is limited sometimes.
- The multi-interval scheme provides us sufficient flexibility to handle ODEs, e.g., we may use geometrically refined steps and linearly increasing degree vectors to resolve the singular behavior of the solution.

Numerical illustrations also show that the suggested algorithms are particularly attractive for ODEs with stiff behaviors, oscillating solutions, steep gradient solutions and long time calculations.

We highlight the main differences between our strategy and the existing ones as follows.

- We collocate the numerical scheme at Chebyshev-Gauss points, and analyze the convergence of the single interval scheme. The nodes and weights of Chebyshev-Gauss quadratures are given explicitly, avoiding the potential loss of accuracy (compared with Legendre and Laguerre quadratures). Particularly, the algorithm can be implemented efficiently by using fast Chebyshev transform. The existing work on spectral collocation methods [18–21, 45] studied the Legendre and Laguerre collocation schemes.
- We use the Chebyshev expansions in each sub-step (known to be much stable than the usual Lagrange approach [38]), which lead to quite neat implementation through manipulating the expansion coefficients (see (2.24) below). The existing work on spectral collocation methods [18–21, 45] considered the Legendre and Laguerre expansions in each sub-step, but did not establish the relationships of the expansion coefficients.

The paper is organized as follows. In the next section, we present and analyze the single interval Chebyshev-Gauss collocation method, and provide some numerical results to justify our theoretical analysis. In Section 3, we describe the multi-interval Chebyshev-Gauss collocation method, the convergence is illustrated numerically. The final section is for some concluding discussions.

We end this section with some notations to be used throughout the paper: