Journal of Computational Mathematics
Vol.33, No.1, 2015, 33–58.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1407-m4486

ON PRECONDITIONING OF INCOMPRESSIBLE NON-NEWTONIAN FLOW PROBLEMS*

Xin He
Delft Institute of Applied Mathematics, Delft University of Technology, Netherlands
Email: X.He-1@tudelft.nl
Maya Neytcheva
Department of Information Technology, Uppsala University, Uppsala, Sweden
Email: maya.neytcheva@it.uu.se
Cornelis Vuik
Delft Institute of Applied Mathematics, Delft University of Technology, Netherlands
Email: c.vuik@tudelft.nl

Abstract

This paper deals with fast and reliable numerical solution methods for the incompressible non-Newtonian Navier-Stokes equations. To handle the nonlinearity of the governing equations, the Picard and Newton methods are used to linearize these coupled partial differential equations. For space discretization we use the finite element method and utilize the two-by-two block structure of the matrices in the arising algebraic systems of equations. The Krylov subspace iterative methods are chosen to solve the linearized discrete systems and the development of computationally and numerically efficient preconditioners for the two-by-two block matrices is the main concern in this paper. In non-Newtonian flows, the viscosity is not constant and its variation is an important factor that effects the performance of some already known preconditioning techniques. In this paper we examine the performance of several preconditioners for variable viscosity applications, and improve them further to be robust with respect to variations in viscosity.

Mathematics subject classification: 65F10, 65F08, 65N30.
Key words: non-Newtonian flows, Navier-Stokes equations, Two-by-two block systems, Krylov subspace methods, Preconditioners.

1. Introduction

Numerical algorithms for incompressible non-Newtonian flows have been intensively studied in the past decades. In non-Newtonian flows the viscosity is not constant and may depend on the velocity, which leads to two nonlinear sources in the governing equations, i.e., the diffusion and convection terms. Due to this, the numerical simulation of the incompressible non-Newtonian flows is more complicated than Newtonian flows, where the viscosity is constant and the only source of nonlinearity in the governing equations is the convection term.

A common approach to solve a nonlinear problem is converting it into a linearized problem, computing the updates of the unknowns by solving the linearized problem and iteratively converging to the true nonlinear solutions. If we consider linearization of both the two nonlinear terms, the variable viscosity Oseen-type problem arises. Ignoring the linearization of the convection term leads to the variable viscosity Stokes-type problem, e.g. [16, 30]. The benefit

* Received January 24, 2014 / Revised version received May 21, 2014 / Accepted July 13, 2014 / Published online December 1, 2014 /

of solving the Stokes-type problem is that efficient solution algorithms are easier to construct compared to the Oseen-type problem. On the other hand, it may take more nonlinear iterations to converge for the Stokes-type problem, typically when the convection is relatively dominant. For each type problem, two well-known linearization methods are used, namely, Picard and Newton iterations. To avoid possible slow convergence rate of Picard iterations and the possibly narrow convergence region of Newton iterations, in this paper a combination of these two iteration methods is utilised. We first carry out some Picard iterations to obtain a reasonably "good" solution, and then use this solution as an initial guess for the Newton iterations. We show that in this way a fast convergence of the nonlinear iterations can be achieved.

For the variable viscosity Oseen- and Stokes-type problems with Picard and Newton iterations, the finite element discretization of the linearized problems results in discrete linear systems of two-by-two block form. Solving the linear systems is the most time-consuming task in the numerical simulations. In this paper, Krylov subspace methods with appropriate preconditioners are chosen to solve the arising linear systems. The kernel of this paper is the construction and the analysis of fast and reliable preconditioning techniques for the variable viscosity Oseen- and Stokes-type problems with both Picard and Newton iterations. As far as the authors know, in earlier works, efficient preconditioners for the variable viscosity Oseen- and Stokes-type problems are only studied for Picard iterations, e.g. [16, 18, 30].

In the past decades, the most often used preconditioners for incompressible Navier-Stokes equations are originally proposed and analysed for the constant viscosity cases, c.f., the surveys [7,10] and the books [1,14,32]. Due to their algebraic construction, some of these preconditioners can be straightforwardly utilised for the variable viscosity applications. In this paper we choose the augmented Lagrangian preconditioner for the Oseen-type problem (Section 3) and the block lower-triangular and the SIMPLER preconditioners for the Stokes-type problem (Section 4). As variable viscosity is an important factor, a crucial objective for having a fast and reliable preconditioner in this case is the robustness with respect to those variations. In order to fully achieve this objective, we modify the above mentioned preconditioners and also propose some computational improvements. The comparison between the targeted preconditioners and the efficiency of the Oseen- and Stokes-type problems are illustrated in Section 5. Conclusions and future work are outlined in Section 6.

2. Problem Formulation and Linearization

In this paper, we assume that the velocity $\mathbf{u}$ and the pressure p satisfy the following generalized stationary incompressible Navier-Stokes equations:

$$\begin{aligned} -\nabla\cdot(2\nu(D_{\mathrm{II}}(\mathbf{u}),p)\mathbf{D}\mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0, \quad & \text{in } \Omega \end{aligned} \tag{2.1}$$

with boundary conditions given by

$$\begin{aligned} \mathbf{u} = \mathbf{g}, \quad & \text{on } \partial\Omega_D \\ \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{n}p = 0. \quad & \text{on } \partial\Omega_N \end{aligned}$$

Here Ω is a bounded and connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, and $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is its boundary, where $\partial\Omega_D$ and $\partial\Omega_N$ denote the parts of the boundary where Dirichlet and Neumann boundary conditions for $\mathbf{u}$ are imposed, respectively. The terms $\mathbf{f} : \Omega \to \mathbb{R}^d$ and $\mathbf{g}$