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GRADED MESHES FOR HIGHER ORDER FEM*

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Abstract

A singularly perturbed one-dimensional convection-diffusion problem is solved numerically by the finite element method based on higher order polynomials. Numerical solutions are obtained using S-type meshes with special emphasis on meshes which are graded (based on a mesh generating function) in the fine mesh region. Error estimates in the ε-weighted energy norm are proved. We derive an 'optimal' mesh generating function in order to minimize the constant in the error estimate. Two layer-adapted meshes defined by a recursive formulae in the fine mesh region are also considered and a new technique for proving error estimates for these meshes is presented. The aim of the paper is to emphasize the importance of using optimal meshes for higher order finite element methods. Numerical experiments support all theoretical results.

Mathematics subject classification: 65L11, 65L50, 65L60, 65L70.
Key words: Singular perturbation, Boundary value problem, Layer-adapted meshes, Graded meshes, Finite element method.

1. Introduction

We consider the singularly perturbed boundary value problem

$$-\varepsilon u'' - b(x)u' + c(x)u = f(x), \quad u(0) = u(1) = 0, \tag{1.1}$$

where $0 < \varepsilon << 1$, $b(x) > 1$ for $x \in [0,1]$ (we avoid some additional parameter β from the assumption $b(x) > \beta > 0$). Let the functions b, c, f be sufficiently smooth. Because (1.1) is characterized by a boundary layer at $x = 0$, the Galerkin finite element method on a standard mesh works unsatisfactory. Therefore 20 years ago exponentialy fitted splines attracted many researchers (see Section 2.2.5 in Part I of [10]). But later it turned out that standard splines on layer-adapted meshes were better. Today layer adapted meshes are the main ingredient in handling problems with boundary or interior layers.

The first, uniformly with respect to ε, convergence result for *linear* finite elements on a Shishkin mesh in the ε-weighted H^1- norm was proved in [12] (for the precise definition of

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norms and meshes see Section 2). That mesh is equidistant and fine in a small interval $[0, \tau]$, where the choice of $\tau = \tau_0 \varepsilon \ln N$ is important, and equidistant but coarse in the large subinterval $[\tau, 1]$ with a mesh size of order $O(N^{-1})$. The resulting error estimate

$$\|u - u^N\|_\varepsilon \leq C(N^{-1} \ln N)$$

(by C we denote a generic constant independent of ε) is not optimal due to the logarithmic factor.

In [7] the authors introduced the class of S-type meshes, where in the fine subinterval $[0, \tau]$ the mesh is not necessarily equidistant but graded based on a mesh generating function. Then, for certain mesh generating functions the optimal estimate

$$\|u - u^N\|_\varepsilon \leq CN^{-1}$$

follows. While for linear elements the influence of the logarithmic factor is not that large, the expressions $(N^{-1} \ln N)^k$ and N^{-k} differ significantly for larger k. Therefore, for higher order finite element methods it is important to use a graded mesh and not a piecewise equidistant mesh. In Section 2 we generalize the results of [7] and prove for finite elements based on polynomials of order k

$$\|u - u^N\|_\varepsilon \leq C(N^{-1} \max |\psi'|)^k.$$

Here ψ is the so-called mesh characterizing function. For some meshes (for instance, Bakhvalov-Shishkin or Vulanović meshes) it follows

$$\|u - u^N\|_\varepsilon \leq CN^{-k}.$$

In Section 3 we use some heuristic arguments to optimize the mesh generating function, i.e. to minimize the constant in the error estimate. Although the analysis is incomplete the numerical results in Section 5 confirm the nice properties of the 'optimal' mesh generating function.

Layer-adapted meshes can also be defined by a *recursive* formula. Examples are the Gartland-Shishkin meshes [9] or meshes analysed in [2]. We modify these meshes and use the recursive formula only in the subinterval $[0, \tilde{\tau}]$, where $\tilde{\tau}$ is the smallest mesh point not smaller than τ generated by the recursive formula for the mesh points. In Section 4 we present a new technique to prove error estimates for these meshes. Finally, in Section 5 we present a detailed numerical study for the comparison of all these meshes, especially for the polynomials of degree $k = 2, 3$. The numerical results confirm the obtained estimates and demonstrate the important role of Bakhvalov-Shiskin and Gartland-Shishkin meshes.

2. S-type Meshes Based on Mesh Generating Functions

It is well known that for the problem (1.1) with smooth data the following solution decomposition into a smooth part S, and a layer part E exists [10]:

$$u = S + E,$$

with

$$|S^{(k)}| \leq C, \quad |E^{(k)}| \leq C\varepsilon^{-k} e^{-x/\varepsilon} \quad k = 0, 1, \dots, q \quad \text{(for any prescribed } q). \tag{2.1}$$

On a given mesh $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$, we denote by V^N the finite element space of continuous piecewise polynomials of degree k. The weak formulation of (1.1) is based on the bilinear form

$$a(w, v) := \varepsilon(w', v') + (-bw' + cw, v) \tag{2.2}$$