Journal of Computational Mathematics
Vol.32, No.6, 2014, 696–720.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1405-m4417

ON SOLVABILITY AND WAVEFORM RELAXATION METHODS FOR LINEAR VARIABLE-COEFFICIENT DIFFERENTIAL-ALGEBRAIC EQUATIONS *

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Abstract

This paper is concerned with the solvability and waveform relaxation methods of linear variable-coefficient differential-algebraic equations (DAEs). Most of the previous works have been focused on linear variable-coefficient DAEs with smooth coefficients and data, yet no results related to the convergence rate of the corresponding waveform relaxation methods has been obtained. In this paper, we develope the solvability theory for the linear variable-coefficient DAEs on Legesgue square-integrable function space in both traditional and least squares senses, and determine the convergence rate of the waveform relaxation methods for solving linear variable-coefficient DAEs.

Mathematics subject classification: 65F10, 65L20, 65L80, 65R10, 65R20.
Key words: Differential-algebraic equations, Integral operator, Fourier transform, Waveform relaxation method.

1. Introduction

Consider the linear variable-coefficient differential-algebraic equations (also called singular systems, descriptor systems, implicit or constrained systems)

$$B(t)\dot{x}(t)+A(t)x(t)=f(t),\ \forall\ t\in\Omega\subseteq\mathbb{R}, \tag{1.1}$$

where $B(t)$ and $A(t)$ are $r\times r$ complex matrix-valued coefficients, $x(t)$ and $f(t)$ are r-dimensional complex vector-valued functions, and $B(t)$ is identically singular. The theory for linear constant-coefficient DAEs has become well developed over the last 30 years; see [7,8]. However, progress on the linear variable-coefficient DAEs has been less complete. The main idea for studying the existence and uniqueness of the solution of the DAEs (1.1) is utilizing coordinate changes to reduce the DAEs (1.1) to the so-called standard canonical form [20],

$$\begin{cases} \dot{y}(t)+C(t)y(t)=g(t), \\ N(t)\dot{z}(t)+z(t)=h(t), \end{cases}$$

where $N(t)$ is strictly lower triangular. This canonical form approach was continued in [16, 25]. However, examples in [6, 9, 10] showed that not all solvable systems could be put into this canonical form. The stability of the DAEs (1.1) is often studied in the sense of continuous dependence of the solution on initial value; see [5, 12]. The derivation of numerical methods for

* Received July 30, 2013 / Revised version received February 24, 2014 / Accepted May 26, 2014 / Published online September 3, 2014 /

the DAEs (1.1) is closely related to determining all or part of the completion of the original vector field defined by the DAEs (1.1), i.e., determining the following ODEs derived from the DAEs (1.1),

$$\dot{x}(t) = Q(t)x(t) + \sum_{i=0}^{\ell} R_i(t) f^{(i)}(t). \tag{1.2}$$

The approximate solution computed by numerical methods related to the completion can be essentially arbitrary off the solution manifold if no restriction is placed on these numerical methods.

In the solvability theory of the DAEs (1.1) related to the standard canonical form, the variable-coefficients $B(t)$ and $A(t)$ are assumed $2r$-times differentiable, data $f(t)$ is assumed at least r-times differentiable, and if $f(t)$ is assumed m-times differentiable, $r \leq m \leq 2r$, the solution $x(t)$ needs to be assumed $(m - r + 1)$-times differentiable. In this paper, we study the solvability in a completely new way under weaker smoothness requirements, i.e., coefficients $B(t)$ and $A(t)$, data $f(t)$, and solution $x(t)$ are Lebesgue square-integrable. By using the Fourier transform, the linear variable-coefficient DAEs (1.1) are transformed into the Fredholm integral equation of the first kind, which is studied by taking advantage of the theory of compact operator. The solvability of the linear variable-coefficient DAEs (1.1) is discussed in both traditional and least squares senses. We eventually find the explicit expression of the solution and the sufficient conditions to guarantee existence and uniqueness of the solution. Furthermore, we are concerned with the stability of the DAEs (1.1) in the sense of the continuous dependence of the solution on data $f(t)$ rather than the initial value.

For numerical solution of the linear variable-coefficient DAEs (1.1), we concentrate on the waveform relaxation methods instead of the methods based on computing the completion (1.2). The waveform relaxation methods are powerful solvers for numerically computing the solution of the DAEs on both sequential and parallel computers; see [1–3, 19, 24, 26]. The basic idea of the above iteration methods is to apply the relaxation technique directly to the DAEs. Therefore, these methods can be regarded as natural extensions of the classical relaxation methods for solving systems of algebraic equations with iterating space changing from $\mathbb{R}^r$ to the function space. The waveform relaxation methods were first introduced by Lelarasmee in [18] for simulating the behavior of very large-scale electrical networks. Lelarasmee proved that the waveform relaxation method is convergent as long as the splitting function of the system is Lipschitz continuous. Later, most of the effort has been made on the expansions and applications of this theory; see [15]. However, there is no precise description about the convergence rate until Miekkala first obtained the convergence rate of the waveform relaxation method of linear constant-coefficient ODEs and DAEs; see [21, 22]. Then, Janssen and Vandewalle studied the convergence rate of different SOR acceleration schemes of the waveform relaxation method for linear constant-coefficient ODEs in [14]. In addition, Pan and Bai further studied the monotone convergence rate of the waveform relaxation methods for linear constant-coefficient ODEs in [23]. The latest result related to the convergence rate of the waveform relaxation method of linear constant-coefficient DAEs is given by Bai and Yang in [4]. In this paper, we study the waveform relaxation methods for solving linear variable-coefficient DAEs in both traditional and least squares sense. The explicit iteration form of the waveform relaxation methods is first proposed. Moreover, we find the spectral radius of the iteration operator, i.e., convergence rate, of the waveform relaxation methods.

The paper is organized as follows. The theories of integral operators are generalized from