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UNIFORMLY CONVERGENT NONCONFORMING ELEMENT FOR 3-D FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM*

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Abstract

In this paper, using a bubble function, we construct a cuboid element to solve the fourth order elliptic singular perturbation problem in three dimensions. We prove that the nonconforming C^0-cuboid element converges in the energy norm uniformly with respect to the perturbation parameter.

Mathematics subject classification: 65N12, 65N30.
Key words: Nonconforming finite element, Singular perturbation problem, Uniform error estimates.

1. Introduction

Let $\Omega \subset R^3$ be a bounded polyhedral domain with boundary $\partial\Omega$. For $f \in L^2(\Omega)$, we consider finite element methods for the following boundary value problem of fourth order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Δ is the standard Laplace operator, $\partial u/\partial n$ denotes the outer normal derivative on $\partial\Omega$ and ε is a small real parameter with $0 < \varepsilon \leq 1$. This problem can be considered a gross simplification of the stationary Cahn-Hilliard equation with ε being the length of the transition region of phase separation. In particular, we are interested in the regime when ε tends to zero. Obviously, if ε approaches zero, the differential Eq. (1.1) formally degenerates to Poisson's equation.

For ε=1, that is, the usual fourth order elliptic equation, many works have been done. When a conforming finite element is used, it should consist of piecewise polynomials that are globally continuously differentiable (C^1). Such elements require polynomials of high degree and even in two dimensions are not easy to construct. To lower the polynomial degree, some macroelements were created on triangle grids, see e.g., [1, 2]. Recently, a macro type of biquadratic C^1 finite element was constructed on rectangle grids [3, 4], which is a rectangular version of the (C^1) Powell-Sabin element [1]. On the other hand, many lower degree nonconforming elements in the two and three dimensional cases have been constructed and used in practice.

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For the fourth order elliptic singular perturbation problem (1.1), the Morley element is a nature choice for the biharmonic operator since it has the least number of degrees of freedom on each element for fourth order problems [5]. Unfortunately, this element is divergent for general second order elliptic problems [2,6-8], so we can not get the uniformly convergent result as $\varepsilon \to 0$ as was shown in [6]. As a result, in order to obtain robust schemes, either the formulation of the Morley element method must be modified or the element itself must be changed, and several variants of the Morley element method have been presented [6, 9, 10].

In the two-dimensional case, a nonconforming C^0 triangular element was constructed in [6], by enriching second degree polynomials with cubic bubble function. A modified triangular Morley element and a modified rectangular Morley element were presented in [9]. In that paper, the authors used the original Morley element and changed the discrete variational problem. An C^0 rectangular element was constructed in [10]. A nine parameter non-C^0 triangular element and a twelve parameter non-C^0 rectangular element were proposed in [11] by the double set parameter technique. Later, by the same technique, a nine parameter C^0 triangular element was analyzed in [12] and a non-C^0 rectangular element was constructed in [13], but the later paper was solving problem (1.1) but with boundary conditions $u = \partial^2 u/\partial n^2 = 0$. All of these nonconforming elements were proved to be uniformly convergent.

In the three-dimensional case. A nonconforming non-C^0 tetrahedral element was constructed and analyzed in [14] by the similar way used in [9], and a nonconforming C^0 tetrahedral element was constructed in [15]. Recently, a nonconforming C^0 tetrahedral element was constructed in [16] by Nitsche's method. In this paper, we introduce an C^0 cuboid element, which was constructed in [17] by us, but the error estimate was valid only for $\varepsilon = 1$. Here, we prove that the element is robust with respect to the perturbation parameter and uniforming convergent to order $h^{1/2}$. Moreover, besides the theoretical interest, our new finite element method is expected to be useful in the computation of the Cahn-Hilliard equation.

The rest of this paper is organized as follows. In the following section, we list some notations and two basic preliminaries. Next, we give detailed descriptions of the cuboid element. Finally, we prove the element is uniformly convergent in ε for the fourth order elliptic singular perturbation equation.

2. Preliminaries

For a nonnegative integer m, we shall use the standard notation $H^m(\Omega)$ to denote the Sobolev space of functions with partial derivatives up to m in $L^2(\Omega)$. The corresponding norm and semi-norm are denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. The space $H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{m,\Omega}$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$. P_k is the polynomial space of degree not greater than k and Q_k is the polynomial space of degree in each coordinate not greater than k.

Define

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} v dx, \quad \forall u, v \in H^2(\Omega), \tag{2.1}$$

$$b(u,v) = \int_\Omega \sum_{i=1}^{3} \partial_i u \partial_i v dx, \qquad \forall u, v \in H^1(\Omega). \tag{2.2}$$

The weak form of (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = f(v), \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$