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ON RESIDUAL-BASED A POSTERIORI ERROR ESTIMATORS FOR LOWEST-ORDER RAVIART-THOMAS ELEMENT APPROXIMATION TO CONVECTION-DIFFUSION-REACTION EQUATIONS*

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Abstract

A new technique of residual-type a posteriori error analysis is developed for the lowest-order Raviart-Thomas mixed finite element discretizations of convection-diffusion-reaction equations in two- or three-dimension. Both centered mixed scheme and upwind-weighted mixed scheme are considered. The a posteriori error estimators, derived for the stress variable error plus scalar displacement error in L^2-norm, can be directly computed with the solutions of the mixed schemes without any additional cost, and are proven to be reliable. Local efficiency dependent on local variations in coefficients is obtained without any saturation assumption, and holds from the cases where convection or reaction is not present to convection- or reaction-dominated problems. The main tools of the analysis are the postprocessed approximation of scalar displacement, abstract error estimates, and the property of modified Oswald interpolation. Numerical experiments are carried out to support our theoretical results and to show the competitive behavior of the proposed posteriori error estimates.

Mathematics subject classification: 65N15, 65N30, 76S05.
Key words: Convection-diffusion-reaction equation, Centered mixed scheme, Upwind-weighted mixed scheme, Postprocessed approximation, A posteriori error estimators.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain in $\mathbb{R}^d, d = 2$ or 3. We consider the following homogeneous Dirichlet boundary value problem for the convection-diffusion-reaction equations:

$$\begin{cases} -\nabla \cdot (S\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $S \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ denotes an inhomogeneous and anisotropic diffusion-dispersion tensor, $\mathbf{w}$ is a (dominating) velocity field, r a reaction function, f a source term. The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions. This type of equations arise in many chemical and biological settings. For instance,

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in hydrology these equations govern the transport and degradation of adsorbing contaminants and microbe-nutrient systems in groundwater.

Reliable and efficient a posteriori error estimators are an indispensable tool for adaptive algorithms. For second-order elliptic problems without convection term, the theory of a posteriori error estimation has reached a degree of maturity for finite elements of conforming, nonconforming and mixed types; see [1-9, 11-14, 18, 20, 22-23, 27, 31-34] and the references therein. For convection-diffusion(-reaction) problems, on the contrary, the theory is still under development.

The mathematical analysis of robustness of a-posteriori estimators for convection-diffusion-reaction equations was first addressed by *Verfürth* [35] in the singular perturbation case, namely $S = \varepsilon I$ with I the identical matrix and $0 < \varepsilon \ll 1$. The proposed estimators for the standard Galerkin approximation and the SUPG disctetization give global upper and local lower bounds on the error measured in the energy norm, and are robust when the Péclet number is not large. In [36] *Verfürth* improved the results of [35] in the sense that the derived estimates are fully robust with respect to convection dominance and uniform with respect to the size of the zero-order reaction term. *Sangalli* [31] developed an a posteriori estimator for the residual-free bubbles methods applied to convection-diffusion problems. Later he presented a residual-based a posteriori estimator for the one-dimensional convection-diffusion-reaction model problem [32]. In [23,24] *Kunert* carried out a posteriori error estimation for the SUPG approach to a singularly perturbed convection- or reaction-diffusion problem on anisotropic meshes. One may also refer to [26, 27] for a posteriori error estimation in the framework of finite volume approximations.

For the convection-diffusion-reaction model (1.1), following an idea of postprocessing in [25] *Vohralík* [37] established residual a posteriori error estimates for lowest-order Raviart-Thomas mixed finite element discretizations on simplicial meshes. Global upper bounds and local lower bounds for the postprocessed approximation error, $p - \tilde{p}_h$, in the energy norm were derived with $\tilde{p}_h$ the postprocessed approximation to the finite element solution p_h, and the local efficiency of the estimators was shown to depend only on local variations in the coefficients and on the local Péclet number. Moreover, the developed general framework allows for asymptotic exactness and full robustness with respect to inhomogeneities and anisotropies.

In this paper, we develop a new technique for residual-based a posteriori estimation of the lowest-order Raviart-Thomas mixed finite element schemes (centered mixed scheme and upwind-mixed scheme) over both the stress error, $\mathbf{u} - \mathbf{u}_h$, and the displacement error, $p - p_h$, of the mixed finite element solutions $(\mathbf{u}_h, p_h)$ for the problem (1.1) with $\mathbf{u} := -S\nabla p$. Local efficiency dependent only on local variations of the coefficients is obtained without any saturation assumption, holds for the convection or reaction dominated equations. Especially for the centered mixed scheme in the singular perturbation case, the proposed estimator yields global upper and local lower bounds which differ by multiplicative constants depending only on the shape regularity parameter and the local mesh-Péclet number. Compared with the standard analysis to the diffusion equations, our analysis avoids, by using the postprocessed approximation $\tilde{p}_h$ as a transition, Helmholtz decomposition of stress variables and dual arguments of displacement error in L^2-norm, and then does not need any weak regularity assumption on the diffusion-dispersion tensor. We note that although being employed in our analysis, the post-processed displacement approximation and its modified Oswald interpolation are not involved in our estimators.

The rest of this paper is organized as follows. In Section 2 we give notations, assumptions of data, and the weak problem. We introduce in Section 3 the mixed finite element schemes