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A NEW PRECONDITIONING STRATEGY FOR SOLVING A CLASS OF TIME-DEPENDENT PDE-CONSTRAINED OPTIMIZATION PROBLEMS*

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Abstract

In this paper, by exploiting the special block and sparse structure of the coefficient matrix, we present a new preconditioning strategy for solving large sparse linear systems arising in the time-dependent distributed control problem involving the heat equation with two different functions. First a natural order-reduction is performed, and then the reduced-order linear system of equations is solved by the preconditioned MINRES algorithm with a new preconditioning techniques. The spectral properties of the preconditioned matrix are analyzed. Numerical results demonstrate that the preconditioning strategy for solving the large sparse systems discretized from the time-dependent problems is more effective for a wide range of mesh sizes and the value of the regularization parameter.

Mathematics subject classification: 65F08, 65F10, 65F50, 49M25, 49K20, 65N22.
Key words: PDE-constrained optimization, Reduced linear system of equations, Preconditioning, Saddle point problem, Krylov subspace methods.

1. Introduction

In this paper, we focus on preconditioned iterative methods for solving the large linear system arising in the time-dependent distributed control problem involving the heat equation. Specifically, we consider the following distributed control of the heat equations:

$$\min_{y,u} J(y,u), \qquad \text{subject to} \begin{cases} \frac{\partial y}{\partial t} - \nabla^2 y = u, & \text{for } (x,t) \in \Omega \times (0,T), \\ y = f, & \text{on } \partial\Omega \times (0,T), \\ y = y_0, & \text{at } t = 0, \end{cases} \tag{1.1}$$

for certain functional $J(y,u)$, where f and y_0 depend maybe on x but not on t. Two target functionals to be considered in this paper are

$$J_1(y,u) = \frac{1}{2}\int_0^T\int_\Omega (y(x,t) - \bar{y}(x,t))^2 d\Omega dt + \frac{\beta}{2}\int_0^T\int_{\partial\Omega} (u(x,t))^2 d\Omega dt,$$

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and

$$J_2(y,u)=\frac{1}{2}\int_{\Omega}(y(x,T)-\bar{y}(x))^2d\Omega+\frac{\beta}{2}\int_0^T\int_{\partial\Omega}(u(x,t))^2d\Omega dt.$$

Here y, u, $\bar{y}$ and p are vectors corresponding to the state, control, desired state and adjoint at all time steps $1,2,\cdots,N_t$, respectively, and β is the regularization parameter.

By the way of $J(y,u)=J_1(y,u)$ being applied the discretize-then-optimize approach ([20,34]), which discretizes this problem with equal-order finite element basis functions for y,u, and the adjoint variable p, results in the linear system ([34])

$$\begin{pmatrix}\tau\mathcal{M}_{1/2} & 0 & \mathcal{K}^T\\ 0 & \beta\tau\mathcal{M}_{1/2} & -\tau\mathcal{M}_{1,1}\\ \mathcal{K} & -\tau\mathcal{M}_{1,1} & 0\end{pmatrix}\begin{pmatrix}y\\u\\p\end{pmatrix}=\begin{pmatrix}\tau\mathcal{M}_{1,1}\bar{y}\\0\\d\end{pmatrix}, \tag{1.2}$$

where $\mathcal{K}$, $\mathcal{M}_{1/2}$ and $\mathcal{M}_{1,1}$ are all matrices in $\mathbb{R}^{(nN_t)\times(nN_t)}$, and

$$\mathcal{K}=\begin{pmatrix}M+\tau K & & & & \\ -M & M+\tau K & & & \\ & \ddots & \ddots & & \\ & & -M & M+\tau K & \\ & & & -M & M+\tau K\end{pmatrix},$$

$$\mathcal{M}_{1/2}=\begin{pmatrix}\frac{1}{2}M & & & & \\ & M & & & \\ & & \ddots & & \\ & & & M & \\ & & & & \frac{1}{2}M\end{pmatrix},\quad d=\begin{pmatrix}\mathcal{M}_{1,1}y_0+c\\ c\\ \vdots\\ c\\ c\end{pmatrix}.$$

Here and in the following, I denotes the identity matrix. Denote by

$$\mathcal{M}_2:=\begin{pmatrix}2M & & & & \\ & M & & & \\ & & \ddots & & \\ & & & M & \\ & & & & 2M\end{pmatrix},\quad \mathcal{I}_s:=\begin{pmatrix}sI & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & sI\end{pmatrix},$$

$$\mathcal{M}_{\gamma,\delta}:=\begin{pmatrix}\gamma M & & & & \\ & \gamma M & & & \\ & & \ddots & & \\ & & & \gamma M & \\ & & & & \delta M\end{pmatrix}\in\mathbb{R}^{(nN_t)\times(nN_t)},$$

where $s=\frac{1}{2}$ or 2. In the above, N_t is the number of time steps of (constant) size τ used to discretize the PDEs, c the boundary conditions of the PDEs and M a finite element mass matrix and K a stiffness matrix on Ω, which are of the dimension $n\times n$ with n being the degrees of freedom of the finite element approximation.

If $J(y,u)=J_1(y,u)$ is alternatively used in the optimize-then-discretize approach, the linear