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AN IMPROVED NON-TRADITIONAL FINITE ELEMENT FORMULATION FOR SOLVING THE ELLIPTIC INTERFACE PROBLEMS*

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Abstract

We propose a non-traditional finite element method with non-body-fitting grids to solve the matrix coefficient elliptic equations with sharp-edged interfaces. All possible situations that the interface cuts the grid are considered. Both Dirichlet and Neumann boundary conditions are discussed. The coefficient matrix data can be given only on the grids, rather than an analytical function. Extensive numerical experiments show that this method is second order accurate in the L^∞ norm.

Mathematics subject classification: 65N30.
Key words: Elliptic equation, Sharp-edged interface, Jump condition, Matrix coefficient.

1. Introduction and Formulations

Elliptic interface problems are widely used in a variety of disciplines when there are multi-physics and multi-phase materials, such as in electromagnetics, material science, fluid dynamics and so on.

We consider a rectangular domain $\Omega = (x_{min}, x_{max}) \times (y_{min}, y_{max})$. Γ is an interface prescribed by the zero level-set $\{(x, y) \in \Omega \mid \phi(x, y) = 0\}$ of a level-set function $\phi(x, y)$. The advantage of using the level-set function is to represent interface cut locations on the grids without having to parameterize the interface. The unit normal vector of Γ is $n = \frac{\nabla\phi}{|\nabla\phi|}$ pointing from $\Omega^- = \{(x, y) \in \Omega \mid \phi(x, y) \leq 0\}$ to $\Omega^+ = \{(x, y) \in \Omega \mid \phi(x, y) \geq 0\}$. Consider the problem

$$\begin{aligned}
&-\nabla \cdot (\beta(x)\nabla u(x)) = f(x), && x \in \Omega^{\pm},\\
&[u(x)] = a(x), && x \in \Gamma,\\
&[(\beta(x)\nabla u(x)) \cdot n] = b(x), && x \in \Gamma,\\
&u(x) = g(x), && x \in \partial\Omega,\\
&\text{or } \ \frac{\partial u(x)}{\partial n} = 0, && x \in \partial\Omega,
\end{aligned}$$

where $x = (x_1, ..., x_d)$ denotes the spatial variables and $\bigtriangledown$ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a $d \times d$ matrix that is uniformly elliptic on each disjoint

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subdomain, Ω^- and Ω^+, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface Γ. The right-hand side $f(x)$ is assumed to lie in $L^2(\Omega)$. We have the following jump conditions:

$$\begin{cases} [u](x) \equiv u^+(x) - u^-(x) = a(x), \\ [(\beta \bigtriangledown u) \cdot n]_\Gamma (x) \equiv n \cdot (\beta^+(x) \bigtriangledown u^+(x)) - n \cdot (\beta^-(x) \bigtriangledown u^-(x)) = b(x). \end{cases}$$

We introduce the weak solution by the standard procedure of multiplying by a test function and integrating by parts: for the problem with Dirichlet Boundary Condition,

$$\int_{\Omega^+} \beta \bigtriangledown u \cdot \bigtriangledown \psi + \int_{\Omega^-} \beta \bigtriangledown u \cdot \bigtriangledown \psi = \int_\Omega f\psi - \int_\Gamma b\psi; \tag{1.1}$$

and for the problem with Neumann Boundary Condition,

$$\int_{\Omega^+} \beta \bigtriangledown u \cdot \bigtriangledown \psi + \int_{\Omega^-} \beta \bigtriangledown u \cdot \bigtriangledown \psi + \int_{\partial\Omega} \frac{\partial u}{\partial n}\psi = \int_\Omega f\psi - \int_\Gamma b\psi, \tag{1.2}$$

where ψ is in H_0^1 for equation 1.1 and H^1 for (1.2).

The pioneering work on this topic was done by Peskin in 1977. The method he proposed was called "immersed boundary" method [11, 12]. It uses a numerical approximation of the δ-function, which smears out the solution on a thin finite band around the interface Γ. In [13], the "immersed boundary" method was combined with the level set method, resulting in a first order numerical method that is simple to implement, even in multiple spatial dimensions. However, for both methods, the numerical smearing at the interface forces continuity of the solution at the interface, regardless of the interface condition $[u] = a$, where a might not be zero.

To achieve high order accuracy, a large class of finite difference methods have been proposed. The main idea is to use difference scheme and stencils carefully near the interface to incorporate jump conditions and achieve high order local truncation error using Taylor expansion. Using finite difference scheme typically requires taking high order derivatives of jump conditions and interface in Taylor expansion. Also property of the discretized linear system is hard to analyze for interface problem with general jump condition.

The "immersed interface" method presented in [3] can get second-order accuracy. This method incorporates the interface conditions into the finite difference stencil, provided that neither of the two jump conditions are zero. The corresponding linear system is sparse, but not symmetric or positive definite. Various applications and extensions of the "immersed interface" method are discussed in [6].

In [4], on basis of the "immersed interface" method, a fast iterative method was proposed to solve constant coefficient problems with the interface conditions $[u] = 0$ and $[\beta u_n] \neq 0$. Non-body-fitting Cartesian grids are used, and then associated uniform triangulations are added on. Interfaces are not necessarily aligned with cell boundaries. Numerical evidence shows that this method's conforming version achieves second order accuracy in the L^∞ norm, and higher than first order for its non-conforming version.

Using finite element method developed in [17], elliptic problems with the interface conditions $[u] = 0$ and $[\beta u_n] \neq 0$ can obtain second order accuracy in energy norm and nearly second order accuracy in the L^2 norm. Interfaces are aligned with cell boundaries.

In [9, 10], the solution is extended to a rectangular region by using Fredholm integral equations. The proposed method can deal with interface conditions $[u] \neq 0$ and $[u_n] = 0$ and when Greens function is available. The discrete Laplacian was evaluated using these jump conditions