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UNIFORM QUADRATIC CONVERGENCE OF A MONOTONE WEIGHTED AVERAGE METHOD FOR SEMILINEAR SINGULARLY PERTURBED PARABOLIC PROBLEMS*

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Abstract

This paper deals with a monotone weighted average iterative method for solving semilinear singularly perturbed parabolic problems. Monotone sequences, based on the accelerated monotone iterative method, are constructed for a nonlinear difference scheme which approximates the semilinear parabolic problem. This monotone convergence leads to the existence-uniqueness theorem. An analysis of uniform convergence of the monotone weighted average iterative method to the solutions of the nonlinear difference scheme and continuous problem is given. Numerical experiments are presented.

Mathematics subject classification: 65M06, 65N06.
Key words: Semilinear parabolic problem, Singular perturbation, Weighted average scheme, Monotone iterative method, Uniform convergence.

1. Introduction

In this paper we give a numerical treatment for the semilinear singularly perturbed parabolic problem in the form

$$u_t - \mu^2(u_{xx}+u_{yy}) + f(x,y,t,u) = 0, \quad (x,y,t) \in Q = \omega \times (0,T], \tag{1.1a}$$
$$u(x,y,t) = g(x,y,t), \quad (x,y,t) \in \partial\omega \times (0,T], \tag{1.1b}$$
$$u(x,y,0) = \psi(x,y), \qquad x \in \overline{\omega}, \tag{1.1c}$$

where $\omega = \{0 < x < 1\} \times \{0 < y < 1\}$, μ is a small positive parameter, $\partial\omega$ is the boundary of ω, the functions f, g and ψ are smooth in their respective domains, and f satisfies the constraint

$$f_u \geq 0, \quad (x,y,t,u) \in \overline{Q} \times (-\infty,\infty), \quad (f_u = \partial f/\partial u). \tag{1.2}$$

This assumption can always be obtained by a change of variables. Indeed, introduce

$$z(x,y,t) = \exp(-\lambda t)u(x,y,t),$$

where λ is a constant. Now, $z(x,y,t)$ satisfies (1.1) with

$$\varphi = \lambda z + \exp(-\lambda t) f(x,y,t,\exp(\lambda t)z),$$

instead of f, and we have $\varphi_z = \lambda + f_u$. Thus, if $\lambda \geq -\min f_u$, where minimum is taking over the domain from (1.2), we conclude $\varphi_z \geq 0$.

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For $\mu \ll 1$, the problem is singularly perturbed and characterized by boundary layers (regions with rapid change of solutions) near boundary $\partial\omega$ (see [1] for details).

We shall employ the weighted average scheme for approximating the semilinear problem (1.1). This nonlinear ten-point difference scheme can be regarded as taking a weighted average of the explicit and implicit schemes. In order to practically compute the nonlinear weighted average scheme, one requires an efficient numerical method. A fruitful method for solving nonlinear difference schemes is the method of upper and lower solutions and its associated monotone iterations. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete elliptic and parabolic boundary value problems. Most of publications on this topic involve monotone iterative schemes whose rate of convergence is of linear rate (cf. [2–7]) Some accelerated monotone iterative schemes for solving discrete elliptic boundary value problems are given in [8, 9]. An advantage of this accelerated approach is that it leads to sequences which converge either quadratically or nearly quadratically. In [10], an accelerated monotone iterative method for solving discrete parabolic boundary value problems based on the implicit scheme is presented. In [11], a combination of the accelerated monotone iterative method from [10] with monotone Picard iterates is constructed. In [10, 11], the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were omitted.

In this paper, we extend the accelerated monotone iterative method from [10] to the case when on each time level a nonlinear difference scheme based on the weighted average of the explicit and implicit schemes is solved inexactly, and give an analysis of a convergence rate of this monotone iterative method. In [10], it is assumed that a pair of ordered upper and lower solutions is given on each time level, and this pair is used as initial iterates in the accelerated monotone iterative method. Our iterative method combines an explicit construction of initial upper and lower solutions on each time level and the modified accelerated monotone iterative method.

In [3], we investigate uniform convergence properties of the monotone weighted average iterative method applied to solving the semilinear problem (1.1). This monotone method possesses only linear convergence rate. In this paper, we investigate uniform convergence properties of the monotone weighted average iterative method based of the extended accelerated monotone iterative method from [10] . We show that the proposed monotone iterative method possesses quadratic convergence rate.

The structure of the paper as follows. In Section 2, we introduce the nonlinear weighted average scheme for the numerical solution of (1.1). The monotone weighted average iterative method is presented in Section 3. The explicit construction of initial upper and lower solutions is incorporated in the monotone weighted average iterative method. Section 4 deals with existence and uniqueness of the solution to the nonlinear difference scheme. In Section 5, we show that the monotone weighted average iterative method possesses uniform quadratic convergence rate. An analysis of convergence rates of the monotone weighted average iterative method, based of different stopping tests, is given in Section 6. Section 7 deals with uniform convergence of the monotone weighted average iterative method to the continuous problem (1.1). The final Section 8 presents results of numerical experiments where iteration counts are compared between the proposed monotone weighted average iterative method and monotone weighted