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A PRIORI ERROR ESTIMATES OF A FINITE ELEMENT METHOD FOR DISTRIBUTED FLUX RECONSTRUCTION*

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Abstract

This paper is concerned with a priori error estimates of a finite element method for numerical reconstruction of some unknown distributed flux in an inverse heat conduction problem. More precisely, some unknown distributed Neumann data are to be recovered on the interior inaccessible boundary using Dirichlet measurement data on the outer accessible boundary. The main contribution in this work is to establish the some a priori error estimates in terms of the mesh size in the domain and on the accessible/inaccessible boundaries, respectively, for both the temperature u and the adjoint state p under the lowest regularity assumption. It is revealed that the lower bounds of the convergence rates depend on the geometry of the domain. These a priori error estimates are of immense interest by themselves and pave the way for proving the convergence analysis of adaptive techniques applied to a general classes of inverse heat conduction problems. Numerical experiments are presented to verify our theoretical prediction.

Mathematics subject classification: 35R30, 65N30, 65N15.
Key words: Distributed flux, Inverse heat problems, Finite element method, Error estimates.

1. Introduction

Inverse heat conduction problems are frequently encountered in engineering and industrial applications. In this paper we address a priori error estimates of a finite element method for numerical reconstruction of some unknown distributed flux in an inverse heat conduction problem. More precisely, the unknown distributed Neumann data, called *fluxes* in the sequel, are to be determined on the interior inaccessible boundary using Dirichlet measurement data on the outer accessible boundary.

The flux distribution is of paramount practical interest in heat conduction processes, e.g., the real-time monitoring in steel industry [1], the visualization by liquid crystal thermography [9], and estimating the freezing front velocity in the solidification process [24]. But its accurate distribution is rather difficult to obtain on some inaccessible boundary, such as the interior

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boundary of nuclear reactors and steel furnaces. Engineers seek to estimate them from accessible outer boundary measurements, which naturally gives rise to the inverse problem of estimating the distribution of fluxes. The most difficult issue in solving and analyzing the inverse heat problem of recovering the distributed flux lies in the strong instability with respect to the errors in the measurement data, i.e., small perturbation in the measurement data may lead to significant amplification of error in the identified flux. It is well-known that the inverse problem under investigation here is essentially lack of continuous dependance on data, thus ill-posed in Hadamard's sense [12].

In order to achieve a reasonable and practically acceptable numerical reconstruction of the flux, one may have to resort to some regularization techniques to transform the unstable ill-posed heat flux reconstruction process into a stable mathematical one. Several numerical methods have been proposed for the distributed flux reconstruction problem, among which the least-squares formulation [23–25] has received intensive investigations and it has been implemented by means of the boundary integral method [25] and finite element method [23]. Recently, adaptive techniques are introduced in this field for efficiency consideration [16], which, guided by the a posteriori error estimates, refines automatically the mesh to better approximate the local but potentially very important features of the distributed flux, e.g., non-smooth boundaries, discontinuous fluxes, or singular fluxes with spikes or abrupt sign changes. The computational cost is significantly reduced since fine resolution is only necessary in the place that local features lie in. Subsequently, the convergence analysis of the adaptive algorithm is established in [17], which requires an important a priori error estimates to develop an estimate for the quasi-orthogonality of the the discretization error with respect to the energy norm, which explain the coupling relation of errors on two successive meshes.

In this work, we will fill in the gap aforementioned by establishing some important a priori error estimates of finite element solutions to the heat flux reconstruction problems, which are of immense interest in numerical analysis of FEMs. Furthermore, they pave the way for proving the convergence analysis of adaptive techniques applied to a general class of inverse heat conduction problems. The detailed convergence analysis is reported in a separate work [17]. Here we derive the convergence order by using the piecewise linear continuous finite elements in terms of the mesh size by assuming the least regularity of solutions to the PDE system associated with the inverse problem, which is of practical use for reconstructing distributed fluxes of salient features.

The paper is organized as follows. In Section 2, we briefly recall the mathematical description of our flux reconstruction problem by an output least-squares formulation plus some Tikhonov regularization term. Some relevant properties are shortly recalled without proof. In Section 3, the finite element discretization is described in detail for purpose of analysis. In Sections 4 and 5, we derive the a priori energy and L^2 norm error estimates in detail, respectively, under the least assumption of regularity. In Section 6, numerical results of two-dimensional problems on a square and an L-shaped domain are presented to demonstrate the theoretical convergence order from analysis. We conclude the work in Section 7 and point out some future work.

We end this section with some notations and conventions. Throughout the paper we adopt the standard notation $W^{m,p}(D)$ for Sobolev spaces on an open bounded domain D in $\mathbb{R}^d$, and write $H^m(D) = W^{m,2}(D)$ for $p = 2$. The norm and semi-norm of $H^m(D)$ are denoted respectively by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. We use $(\cdot,\cdot)_D$ to denote the inner product in $L^2(D)$. When no confusion is caused, we may simply drop D in the notation $\|\cdot\|_{m,D}$ and $(\cdot,\cdot)_D$. In addition, we will often use c or C to denote generic positive constants which are independent of mesh size h and functions involved.