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ERROR ESTIMATE ON A FULLY DISCRETE LOCAL DISCONTINUOUS GALERKIN METHOD FOR LINEAR CONVECTION-DIFFUSION PROBLEM*

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Abstract

In this paper we present the error estimate for the fully discrete local discontinuous Galerkin algorithm to solve the linear convection-diffusion equation with Dirichlet boundary condition in one dimension. The time is advanced by the third order explicit total variation diminishing Runge-Kutta method under the reasonable temporal-spatial condition as general. The optimal error estimate in both space and time is obtained by aid of the energy technique, if we set the numerical flux and the intermediate boundary condition properly.

Mathematics subject classification: 65M15, 65M60.
Key words: Runge-Kutta, Local discontinuous Galerkin method, Convection-diffusion equation, Error estimate.

1. Introduction

In this paper we shall present the a priori error estimate for one fully discrete algorithm to solve linear convection-diffusion problem with Dirichlet boundary condition. The scheme under consideration in this paper, which is referred to as the LDGRK3 scheme, uses the third order explicit total variation diminishing Runge-Kutta (TVDRK3) time-marching [19] and the local discontinuous Galerkin (LDG) spatial discretization with piecewise polynomials of arbitrary degree $k \geq 1$.

This type of method was introduced by Cockburn and Shu in [10] as an extension to general convection-diffusion problems of the numerical scheme for the compressible Navier-Stokes equation [2], which is a remarkable development from the famous Runge-Kutta discontinuous Galerkin (RKDG) methods for purely hyperbolic problems. After that, this method has been rigorously studied by a number of researchers, for example, for elliptic problem [1, 6], Stokes problem [9], and convection-diffusion problem [7]. For a fairly complete set of references on this method as well as its implementation and applications, please refer to the recent review papers [11, 18] and book [13].

To put our result in proper perspective, let us briefly describe the relevant results available in the current literature. In [10], the semi-discrete LDG method for convection diffusion problems with periodic boundary conditions was considered, and the quasi-optimal error estimate was obtained (namely k-th order accuracy in L^2-norm if the piecewise polynomials of degree k are used). Later, the convergence properties and the optimal error estimate of the hp-version of

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the semi-discrete LDG method has been studied in [7], for convection-diffusion problems with Dirichlet boundary condition.

However, as far as the authors know, there are few analysis for the fully-discrete version of the LDG method. Since we actually care about the convection-dominated case in this paper, we are interesting in the explicit time-marching. We would like to adopt the explicit TVDRK3 time-marching which has a strong stability preserving (SSP) property [12] and high order accuracy in time. Based on the works of [7,22], we will devote to obtaining, for this fully-discrete method, the optimal error estimates in both time and space, for the convection-diffusion problem with Dirichlet boundary condition. In this paper we abandon the second order total variation diminishing Runge-Kutta (TVDRK2) time-marching, since it requires a more strict restriction on the temporal-spatial condition, such as $\tau = \mathcal{O}(h^{4/3})$, for the high-order piecewise polynomials, where h and τ are the maximum cell size and the time step respectively; see [3,20].

The difficulty of analysis in this paper mainly lies in two points. One is how to define nice numerical flux for the general Dirichlet boundary condition other than the periodic boundary condition. It is well known that the numerical flux is an important issue to ensure the success for LDG methods. The numerical flux is easy to implement for periodic boundary conditions, but not for Dirichlet boundary condition. In this paper we would like to follow the idea in [7], and only make some minor modifications on the numerical flux for periodic boundary condition at the boundary points. We want to find a uniform setting for numerical flux to solve out the convection-diffusion problem, in whatever case that the problem is convection-dominated or not.

The other difficulty comes from the boundary treatment at each intermediate stage time. Since the higher order TVDRK3 time marching is made up of the first-order Euler forward time-marching in each stage, incorrect boundary condition treatments may destroy the high accuracy of the LDGRK3 algorithm as that for periodic boundary condition. An important development on this reduction of convergence order has been given by Carpenter and his colleague [4], where some corrections on the intermediate boundary condition is presented for the finite difference methods to solve the hyperbolic equation. In this paper we would like to seek a reasonable treatment on the intermediate boundary condition for convection-diffusion problem from the viewpoint of energy analysis. The strategies presented here are very similar as that given in [4], which are solely based on the physical information of the given boundary condition. Several numerical experiments are also given to show the validation of our strategy on the boundary condition treatment.

The main analysis tool in this paper is the energy technique, which has been used in [20–22] to analyze the fully discrete algorithm of DG method with Runge-Kutta time-marching. This technique has many advantages in the numerical analysis. It does not demand the used mesh in uniform size, and can be extended to problems with varying-coefficients even to the nonlinear problems, it also works well for different types of boundary conditions [14]. Furthermore, it helps us to find out the reasonable and good treatment on the numerical boundary condition [15].

The remainder of this paper is organized as follows. In section 2 we present the LDGRK3 scheme for a model problem. In section 3, we give some preliminaries for the discontinuous finite element space, including the inverse properties and approximation properties for two local Gauss-Radau projections. Then we present some elemental properties of the corresponding LDG spatial discretization. Section 4 is the main body of this paper where the main result on error estimate is presented and proved by energy technique. In this process, the numerical flux and intermediate boundary conditions will be defined well. Several proofs of some basic lemmas