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A SECOND ORDER MODIFIED CHARACTERISTICS VARIATIONAL MULTISCALE FINITE ELEMENT METHOD FOR TIME-DEPENDENT NAVIER-STOKES PROBLEMS*

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Abstract

In this paper, by combining the second order characteristics time discretization with the variational multiscale finite element method in space we get a second order modified characteristics variational multiscale finite element method for the time dependent Navier-Stokes problem. The theoretical analysis shows that the proposed method has a good convergence property. To show the efficiency of the proposed finite element method, we first present some numerical results for analytical solution problems. We then give some numerical results for the lid-driven cavity flow with Re = 5000, 7500 and 10000. We present the numerical results as the time are sufficient long, so that the steady state numerical solutions can be obtained.

Mathematics subject classification: 76D05, 76M10, 65M60, 65M12.
Key words: Modified method of characteristics, Defect-correction finite element method, Navier-Stokes problems, Characteristics-based method, Lid-driven problem.

1. Introduction

Finding efficient numerical methods for the Navier-Stokes equation is a key component in the incompressible flow simulation. There are a few efficient iterative methods for solving the stationary Navier-Stokes equations under some strong uniqueness conditions presented by some authors, see, e.g., [21,23,24]. It is known that variational multiscale (VMS) method is an efficient method for solving the high Reynolds Navier-Stokes equations. The basic idea of the VMS method is to splitting the solution into resolved and unresolved scale, representing the unresolved scales in terms of the resolved scales, and using this representation in the variational equation for the resolved scales. In [26, 27], Hughes and his coworkers presented the VMS method firstly. After then, there are many works devoted to this method, e.g., VMS method for the Navier-Stokes equations [31]; a two-level VMS methods for convection-dominated diffusion problems [32]; VMS methods for turbulent flows [13,34,35]; large-eddy simulation (LES) [28, 29,37]; subgrid-scale models for the incompressible flow [26,46]. There is another class of VMS methods which rely on a three-scale decomposition of the flow field into large, resolved small and unresolved scales [12]. By the difference of the definition of the large-scale projections,

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the VMS methods can be classified into two kinds. In [33, 36], John et al. presented the error analysis of these two kinds of VMS methods for the Navier-Stokes equations.

The characteristics method, in which the hyperbolic part (the temporal and advection term) is treated by a characteristic tracking scheme, is a highly effective method for advection dominated problems. There are many authors devoted to this method, see, e.g., [1, 2, 4, 6, 9, 10, 15, 22, 42, 45]. In [16], Douglas and Russell presented the modified method of characteristics (MMOC) firstly. A characteristics mixed finite element method for advection-dominated transport problems was presented by Arbogast [2]. Russell [41] extended it to nonlinear coupled systems in two and three spacial dimensions. In [39], a detail analysis for the Navier-Stokes equations is provided by Pironneau. He obtained suboptimal convergence rates of the form $\mathcal{O}(h^m + \triangle t + h^{m+1}/\triangle t)$, which is improved by Dawson et al [14]. In [5], Boukir et al. presented a second-order time scheme based on the characteristics method and spatial discretization of finite element type for the incompressible Navier-Stokes equations. An optimal error estimate for the Lagrange-Galerkin mixed finite element approximation of the Navier-Stokes equations was given by Süli in [44]. The second order in time method for linear convection diffusion problems had been given by Ewing and Russell [18]. In [43], a second order MMOC mixed defect-correction finite element method for time dependent Navier-Stokes problems was proposed.

In this paper, we present a second order modified characteristics VMS finite element method for time dependent Navier-Stokes equations. In our method, the hyperbolic part (the temporal and advection term) is treated by a second order characteristic tracking scheme. Then we use VMS based on projection finite element method in space discretization. The error analysis shows that this method has a good convergence property. In order to show the efficiency of the second order MCVMS finite element method, we first present some numerical results of an analytical solution problems. The numerical results show that the convergence rates are $\mathcal{O}(h^3)$ of the L^2-norm for u, $\mathcal{O}(h^2)$ of the semi H^1-norm for u and $\mathcal{O}(h^2)$ of the L^2-norm for p, which agrees very well with our theoretical results by using $P_2 - P_1$ finite element spaces. Then, some numerical results of the lid-driven cavity flow with $Re = 5000$ and 7500 were given, firstly. We present the numerical results as the time are sufficient long enough. By the numerical results, we can see that a steady state numerical solutions of the the time-dependent Navier-Stokes equations were obtained. Meanwhile, the numerical solutions are in good agreement with that of the steady Navier-Stokes equations shown by Ghia et al. [20] and Erturk et al. [17]. At last, we present some numerical results for $Re = 10000$. It shows that the solution is quasi-periodic and has small variations at the monitoring point. The phase portraits of the monitoring points show that the variations in amplitude yield a solution which is quasi-periodic. It is observed from these numerical results that the schemes can results in good accuracy and is highly efficient.

2. Functional Setting of the Navier-Stokes Equations

In this paper, we consider the time-dependent Navier-Stokes(NS) problems

$$\begin{cases} u_t - \nu\triangle u + (u\cdot\nabla)u + \nabla p = f, & x \in \Omega\times[0,T], \\ \nabla\cdot u = 0, & x\in\Omega\times[0,T], \\ u(x,0) = u_0(x), & x\in\Omega, \\ u(x,t) = 0, & x\in\partial\Omega\times[0,T], \end{cases} \tag{2.1}$$