Journal of Computational Mathematics
Vol.31, No.2, 2013, 107–136.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1210-m4094

A MULTI-DOMAIN SPECTRAL IPDG METHOD FOR HELMHOLTZ EQUATION WITH HIGH WAVE NUMBER*

Lunji Song
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email: song@lzu.edu.cn
Jing Zhang
School of Mathematics and Statistics, Huazhong Normal University, Wuhan 430079, China
Email: zjccnu@gmail.com
Li-Lian Wang
Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore
Email: lilian@ntu.edu.sg

Abstract

This paper is concerned with a multi-domain spectral method, based on an interior penalty discontinuous Galerkin (IPDG) formulation, for the exterior Helmholtz problem truncated via an exact circular or spherical Dirichlet-to-Neumann (DtN) boundary condition. An effective iterative approach is proposed to localize the global DtN boundary condition, which facilitates the implementation of multi-domain methods, and the treatment for complex geometry of the scatterers. Under a discontinuous Galerkin formulation, the proposed method allows to use polynomial basis functions of different degree on different subdomains, and more importantly, explicit wave number dependence estimates of the spectral scheme can be derived, which is somehow implausible for a multi-domain continuous Galerkin formulation.

Mathematics subject classification: 65N35, 65E05, 65M70, 41A05, 41A10, 41A25.
Key words: Helmholtz equation, High wavenumber, Global DtN boundary condition, IPDG, Multli-domain spectral method.

1. Introduction

Time harmonic wave propagations appear in many applications, and a variety of situations requires to solve the Helmholtz equation exterior to a bounded obstacle (or scatterer):

$$\begin{cases} -\Delta u - k^2 u = f, & \text{in} \quad \Omega_e := \mathbb{R}^d \setminus B, \\ u = g, & \text{on} \quad \Gamma_B := \partial B, \\ \partial_r u - \mathrm{i}ku = o\big(r^{\frac{1-d}{2}}\big), & \text{as} \quad r \to \infty, \end{cases} \tag{1.1}$$

where $k > 0$ is the wave number, $B \subset \mathbb{R}^d$, $d = 2, 3$ is a scatterer with Lipschitz boundary Γ_B, and the far-field boundary condition is known as the Sommerfeld radiation condition. On the surface of the obstacle B, the Dirichlet boundary condition corresponding to sound soft surface of B is imposed, while the Neumann or Robin boundary condition relative to sound hard or

* Received March 3, 2012 / Revised version received October 9, 2012 / Accepted October 25, 2012 / Published online March 14, 2013 /

impedance surface, respectively, may also be prescribed in practice. In fact, the method to be proposed in this paper works for these possible boundary conditions.

Apparent challenges in solving the exterior Helmholtz equation lie in (i) the domain is unbounded, (ii) the problem is indefinite, and (iii) the solution is highly oscillatory (when the wave number is large) and decays slowly. There is a vast literature devoted to its numerical solutions such as boundary element methods [9], infinite element methods [17], Dirichlet-to-Neumann (DtN) methods [23], perfectly matched layers (PML) [6], among others. In many of these approaches, it is essential to truncate the unbounded domain to a bounded domain by imposing an exact or approximate non-reflecting boundary condition at the outer boundary. Formally, the problem (1.1) reduces to

$$\begin{cases} -\Delta u - k^2 u = f, & \text{in} \quad \Omega := \Omega_R \setminus \bar{B}, \\ u = g, & \text{on} \quad \Gamma_B, \\ \partial_r u + Gu = 0, & \text{on} \quad \Gamma_R, \end{cases} \tag{1.2}$$

where Ω_R is an artificial domain that encloses the bounded scatterer B and contains the support of f, and the Robin boundary involving the operator G describes a typical transparent or non-reflecting boundary condition on the outer boundary Γ_R of Ω_R. For instance, G can be the Dirichlet-to-Neumann operator, which has a series expansion when Ω_R is a separable domain, e.g., disk, ball, ellipse and ellipsoid.

In the past two decades, there has been an intensive research on the finite element discretization of (1.2) in various situations (see, e.g., [3–5, 20–22] and the references therein). It is known that when the wave number k becomes large, the mesh size h should be adapted to k so as to resolve the waves. In two or higher dimensions, under the "rule of thumb" mesh constraints $kh \lesssim 1$, the pollution effect exits for all degrees of approximation and deteriorate the error estimates [20]. Thus, it is important to appreciate how the numerical errors depending on the wave numbers. Babuška et al. [20–22] conducted a rigorous analysis using the (discrete) Green's functions, and Douglas et al. [13] used a different argument due to Schatz [32]. However, these approaches may not be applicable to (1.2) with a slightly complicated setting of boundary conditions or scatterers. Recently, some methodology was developed in [11, 24] (also see [7, 19, 25, 34]) for the *a prior* estimates of the solution of (1.2) in a star-shaped domain Ω.

The spectral method, which is vitally free of dispersive errors, is well-suited for wave simulations. With a proper boundary perturbation technique (or the so-called transformed field expansion) [28], the Helmholtz equation (1.2) with exact DtN boundary condition can be reduced to a sequence of Helmholtz equations in a separable domain, e.g., an annulus and a spherical shell (cf. [14, 29, 30, 33]). Shen and Wang [35] provided a rigorous analysis of the spectral-Galerkin method with explicit dependence of the errors on the wave number for the Helmholtz equation in an annulus or spherical shell with exact DtN boundary condition. The analysis for full coupled spectral-Galerkin and boundary perturbation was conducted in [30]. Indeed, within the domain of applicability of the boundary perturbation method, this approach has proven to be fast and accurate. However, an element method is more desirable, when the scatterer is complex with a large deviation from a "simple" domain.

The purpose of this paper is to propose and analyze a multi-domain spectral interior penalty discontinuous Galerkin (in short, p-IPDG) method, for (1.2) with an exact DtN boundary condition. We advocate a DG formulation for two reasons:

(i) flexibility for general scatters and benefit of p-adaptivity (different orders of polynomials