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BANDED TOEPLITZ PRECONDITIONERS FOR TOEPLITZ MATRICES FROM SINC METHODS *

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Abstract

We give general expressions, analyze algebraic properties and derive eigenvalue bounds for a sequence of Toeplitz matrices associated with the sinc discretizations of various orders of differential operators. We demonstrate that these Toeplitz matrices can be satisfactorily preconditioned by certain banded Toeplitz matrices through showing that the spectra of the preconditioned matrices are uniformly bounded. In particular, we also derive eigenvalue bounds for the banded Toeplitz preconditioners. These results are elementary in constructing high-quality structured preconditioners for the systems of linear equations arising from the sinc discretizations of ordinary and partial differential equations, and are useful in analyzing algebraic properties and deriving eigenvalue bounds for the corresponding preconditioned matrices. Numerical examples are given to show effectiveness of the banded Toeplitz preconditioners.

Mathematics subject classification: 65F10, 65F50.
Key words: Toeplitz matrix, Banded Toeplitz preconditioner, Generating function, Sinc method, Eigenvalue bounds.

1. Introduction

Let $\mathbb{L}^2[-\pi,\pi]$ be the functional space of all quadratically integrable functions defined on the interval $[-\pi,\pi]$. For $f\in\mathbb{L}^2[-\pi,\pi]$, denote by

$$a_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)e^{-\imath k\theta}\mathrm{d}\theta,\quad k\in\mathbb{Z},$$

the Fourier coefficients of f, where $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$ represents the set of all integers and $\imath$ denotes the imaginary unit. For all $n\geq 1$, we write $A_n=(a_{j,k})$ the n-by-n Toeplitz matrix with entries satisfying $a_{j,k}=a_{j-k}$, $1\leq j,k\leq n$. The function f is called the generating function of the sequence of Toeplitz matrices A_n, $n=1,2,\ldots$. Alternatively, we also use $A_n[f]$ to denote the n-by-n Toeplitz matrix generated by the function f. Note that $A_n[f]$ is a non-Hermitian matrix when f is a complex-valued function, and it is a Hermitian matrix when f is a real-valued function. In particular, if f is real-valued and even, then $A_n[f]$ is real symmetric.

Toeplitz systems of linear equations arise in a variety of applications in mathematics and engineering. In particular, when the sinc method is applied to discretize the linear ordinary and partial differential equations, we can often obtain systems of linear equations whose coefficient matrices are combinations of Toeplitz and diagonal matrices; see [1–3, 10, 14]. Hence, it

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is a basic requirement to discuss the algebraic properties of these Toeplitz matrices, construct their effective preconditioners, and derive tight eigenvalue bounds for the corresponding preconditioned matrices. Here, we construct banded Toeplitz preconditioners by making use of trigonometric generating functions, which is an idea first proposed in [6] for Toeplitz matrices with nonnegative generating functions.

In this paper, we first give the general expressions for the Toeplitz matrices associated with the sinc discretizations of various orders of differential operators, and derive the generating functions of these Toeplitz matrices as well as their eigenvalue bounds. According to suitable approximations to the generating functions, we construct banded Toeplitz preconditioners and demonstrate the uniformly bounded property about the spectra of the preconditioned matrices. In particular, we also derive eigenvalue bounds for the banded Toeplitz preconditioners. These results are elementary in constructing high-quality structured preconditioners for the systems of linear equations arising from the sinc discretizations of ordinary and partial differential equations, and are useful in analyzing algebraic properties and deriving eigenvalue bounds for the corresponding preconditioned matrices.

The outline of the paper is as follows. In Section 2, we derive the expression of the Toeplitz matrices from sinc methods and their properties. In Section 3, we construct the banded Toeplitz preconditioners and analyze the eigenvalue bounds for these preconditioners. Some useful bounds are established for the Toeplitz matrices and the banded preconditioners in Section 4. In Section 5, numerical examples are given to show the effectiveness of the banded Toeplitz preconditioners.

2. Toeplitz Matrices

The sinc function is defined as

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad -\infty < t < \infty,$$

and the corresponding sinc basis functions are given by

$$S(j,h)(t) := \frac{\sin[\pi(t - jh)/h]}{\pi(t - jh)/h}, \quad -\infty < t < \infty, \quad j \in \mathbb{Z},$$

where h is the step-size used in sinc methods [14]. The points $t_j = jh$ ($j \in \mathbb{Z}$) are called the sinc-grid points. In sinc methods, we also need to introduce a one-to-one conformal mapping, say, $\phi(x)$, which maps a simply-connected domain onto a strip region.

The n-by-n Toeplitz matrices associated with sinc discretizations of the linear ordinary and partial differential equations are of the form

$$T^{(m)} \equiv \big[\delta_{jk}^{(m)}\big], \qquad j,k \in \mathbb{Z}, \quad m = 0,1,2,\ldots, \tag{2.1}$$

where $\delta_{jk}^{(m)}$ is defined as

$$\delta_{jk}^{(m)} := h^m \frac{\mathrm{d}^m}{\mathrm{d}\phi^m}[S(j,h) \circ \phi(x)]\bigg|_{x=x_k}. \tag{2.2}$$

For example, $T^{(0)} = I$ is the identity matrix, and for $m = 1,2,3,4$ the Toeplitz matrices $T^{(m)}$