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A NUMERICAL METHOD FOR SOLVING THE ELLIPTIC INTERFACE PROBLEMS WITH MULTI-DOMAINS AND TRIPLE JUNCTION POINTS*

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Abstract

Elliptic interface problems with multi-domains and triple junction points have wide applications in engineering and science. However, the corner singularity makes it a challenging problem for most existing methods. An accurate and efficient method is desired. In this paper, an efficient non-traditional finite element method with non-body-fitting grids is proposed to solve the elliptic interface problems with multi-domains and triple junctions. The resulting linear system of equations is positive definite if the matrix coefficients for the elliptic equations in the domains are positive definite. Numerical experiments show that this method is about second order accurate in the L^∞ norm for piecewise smooth solutions. Corner singularity can be handled in a way such that the accuracy does not degenerate. The triple junction is carefully resolved and it does not need to be placed on the grid, giving our method the potential to treat moving interface problems without regenerating mesh.

Mathematics subject classification: 65N06, 65B99.
Key words: Elliptic equations, Non-body-fitting mesh, Finite element method, Triple junction, Jump condition.

1. Introduction

Elliptic interface problems have wide applications in a variety of disciplines. However, designing highly efficient methods for these problems is a difficult job, especially with multi-domains and triple junctions. In the past three decades, much attention has been paid to the numerical solution of elliptic equations with discontinuous coefficients and singular sources on regular Cartesian grids since the pioneering work of Peskin [22] on the first order accurate immersed boundary method. In many applications, particularly for free boundary and moving interface problems, simple Cartesian grids are preferred. In this way, the procedure of generating an unstructured grid can be bypassed, and well developed fast solvers on Cartesian grids can be utilized. With a fixed unstructured grid for moving interface problem, one cannot ensure the triple junction point is a grid point. Without careful treatment, the accuracy is compromised when the triple junction point is not a grid point.

Motivated by the immersed boundary method, to improve accuracy, in [10], the "immersed interface" method (IIM) was presented. This method achieves second order accuracy by incorporating the interface conditions into the finite difference stencil in a way that preserves the interface conditions in both solution and its flux, $[u] \neq 0$ and $[\beta u_n] \neq 0$. The corresponding linear system is sparse, but may not be symmetric or positive definite if there is a jump in the

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coefficient. Various applications and extensions of IIM are discussed in [4, 14]. In [11], a fast iterative method using the augmented IIM was developed for Poisson equations with piecewise constant but discontinuous coefficient. The number of calls to the fast Poisson solver of the method is independent of the jump in the coefficient and the mesh size.

In [12, 13], the immersed finite element methods (IFEM) are developed using non-body-fitted Cartesian meshes for homogeneous jump conditions. The idea is to modify the basis functions so that the homogeneous jump conditions are satisfied. Both non-conforming and conforming IFEM are developed in [13] for 2D problems. Numerical evidence shows that IFEM of the conforming version achieves second order accuracy in the L^∞ norm, and higher than first order for its non-conforming version. In [24], the IFEM is further developed to deal with non-homogeneous jump conditions. The non-conforming immersed finite element methods are also developed for elasticity equations in [5, 24].

In [11], a fast iterative method in conjunction with the "immersed interface" method has been developed for constant coefficient problems with the interface conditions $[u] = 0$ and $[\beta u_n] \neq 0$. Numerical results show that this method's conforming version achieves second order accuracy in the L^∞ norm, and higher than first order for its non-conforming version.

In [7], a non-traditional finite element formulation for solving elliptic equations with smooth or sharp-edged interfaces was proposed with non-body-fitting grids for $[u] \neq 0$ and $[\beta u_n] \neq 0$. It achieved second order accuracy in the L^∞ norm for smooth interfaces and about 0.8th order for sharp-edged interfaces. In [8], the method is modified and improved to close to 2nd order accurate for sharp-edged interfaces, and it is extended to handle general elliptic equations with matrix coefficient and lower order terms. The resulting linear system is non-symmetric but positive definite. In [9], the work was generalized to solve the elasticity interface problems. In [26], the matched interface and boundary (MIB) method was proposed to solve elliptic equations with smooth interfaces. In [25], the MIB method was generalized to treat sharp-edged interfaces. With an elegant treatment, second order accuracy was achieved in the L^∞ norm. Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [1], [19] for two dimensional problems and [20] for three dimensional problems. Another class of methods is the Boundary Condition Capturing Method [16–18].

Although there are many different methods above in the literature for solving the elliptic interface problems with two domains, the elliptic interface problems with multi-domains and triple junctions have not been extensively studied. The new challenges include the treatment of the triple junction point and the added complexity of the problem. In [23], the MIB method is generalized to solve the elliptic interface problems with multi-domains. Numerical evidence shows second order accuracy.

Based on the method in [8], in this paper we propose a numerical method for solving the elliptic problem in multi-domains. We propose an accurate treatment for the triple junction point shown in Figure 3.1. The resulting linear system is positive definite if the matrix coefficients $\beta_i,\ i = 1, 2, 3$ for the elliptic equation in three domains are positive definite. This method is not just a trivial extension of the one in [8], as the triple junction point has to be carefully studied. Numerical results demonstrate about second order accuracy for the method, even with presence of corner singularity. Compared with the existing method [23] for the same setup of the problem, there are two advantages of our method: one is the positive definiteness of the coefficient matrix, the other is the generalization to matrix coefficients β_i instead of scalar coefficients in [23].