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TAILORED FINITE CELL METHOD FOR SOLVING HELMHOLTZ EQUATION IN LAYERED HETEROGENEOUS MEDIUM*

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Abstract

In this paper, we propose a tailored finite cell method for the computation of two-dimensional Helmholtz equation in layered heterogeneous medium. The idea underlying the method is to construct a numerical scheme based on a local approximation of the solution to Helmholtz equation. This provides a computational tool of achieving high accuracy with coarse mesh even for large wave number (high frequency). The stability analysis and error estimates of this method are also proved. We present several numerical results to show its efficiency and accuracy.

Mathematics subject classification: 65N35, 35L10.
Key words: Tailored finite cell method, Helmholtz equation, Heterogeneous media, Sommerfeld condition.

1. Introduction

In this paper, we study the Helmholtz equation in a layered heterogeneous medium

$$\Delta u(\mathbf{x}) + k^2 n^2(x) u(\mathbf{x}) = f(\mathbf{x}), \quad \text{for } \mathbf{x} = (x,y) \in \Omega, \tag{1.1}$$

$$u|_{x=0} = u_0(y), \quad \left(\frac{\partial u}{\partial x} - \mathrm{i} k n(x) u\right)\Big|_{x=R} = 0, \quad \text{for } y \in \mathbb{R}, \tag{1.2}$$

$$\frac{\partial u}{\partial r}(\mathbf{x}) - \mathrm{i} k n(x) u(\mathbf{x}) = o\left(\frac{1}{\sqrt{r}}\right), \quad \text{as} \quad r = |\mathbf{x}| \to +\infty, \tag{1.3}$$

where $\Omega = (0,R) \times \mathbb{R}$, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, k is the wave number, $f \in L^2(\Omega)$, $u_0 \in H^1(\mathbb{R})$. Here the index of refraction $n(x) \in L^\infty(0,R)$ is a piecewise smooth function, which satisfies

$$n_0 \le n(x) \le N_0. \tag{1.4}$$

The boundary value problem of the Helmholtz equation (1.1)–(1.3) arises in many physical fields, for example in seismic imaging where the interior structure of the Earth is layered indeed. Moreover, we can also see similar problems in acoustic wave propagation and electromagnetic wave propagation. The numerical computation of Helmholtz equation with large wave number

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in heterogeneous medium is extremely difficult [2, 22–24] since the mesh size has to be small enough to resolve the wave length. In the last three decades, many scientists have presented efficient methods for this kind of problem, such as the fast multipole method [12], multifrontal method [27], discrete singular convolution method [4], hybrid numerical asymptotic method [11], spectral approximation method [32], element-free Galerkin method [36, 38], geometrical optics-based numerical method [6, 7], *etc.* In general one has the restriction $kh = \mathcal{O}(1)$ for the mesh size h to achieve a satisfactory numerical accuracy. On the other hand, if we use asymptotic method, we usually need to overcome the difficulties about caustics [6, 7, 10, 31].

Tailored finite point method (TFPM) was proposed by Han, Huang and Kellogg for the numerical solutions of singular perturbation problem [18] in 2008. TFPM is different from the typical finite point method [9, 26, 29, 30] which is a development of finite difference method by emphasizing the meshless technique. The main idea underlying the TFPM is to use the exact solution of the local approximate problem to construct the global approximation. Recently, TFPM has been further developed to solve various numerical problems. For example, Han and Huang studied TFPM for the Helmholtz equation in one dimension [13], and obtained the uniform convergence in L^2-norm with respect to the wave number. They also studied TFPM for different kinds of singular perturbation problems [14–16], without any prior knowledge of the boundary/interior layers. This method can provide high accuracy even on the uniform coarse mesh $h \gg \varepsilon$, where ε is the small parameter in the singular perturbation problem. For the interface problem [20], the method produces uniform convergence in energy norm even for the PDEs of mixed type. Later, Shih *et al* proposed a characteristic TFPM and rotated the stencil an angle to keep the grids be a streamline aligned [34, 35], that improved the accuracy on coarse mesh. Furthermore, the method was applied to solve the steady MHD duct flow problems with boundary layers successfully [19]. TFPM also works well for time-dependent problem [21] and fourth-order singular perturbation problem [17]. More related work can be found in two review papers [5, 37] and the references therein. Note that there was also much work about meshless methods for Helmholtz equation [1, 3, 8, 28].

In this paper, we introduce a new approach to construct a discrete scheme for the equation (1.1) based on the former studies [13, 20]. We call the new scheme "*tailored finite cell method*" (TFCM), because it has been tailored to some local properties of the problem in each cell. As we consider the layered medium at here, we will apply our idea after a Fourier transform in y-direction. Hence this is a semi-discrete method designed on the properties of the local approximate problem. The method can achieve high accuracy with relatively cheap computational cost. Especially, we can get the exact solution with fixed points for piecewise linear coefficient for both small and large wave numbers.

2. Tailored Finite Cell Method

In this section, we describe the method in details. To be more precise, in the rest of this paper, we shall assume that the piecewise smooth function $n(x)$ is also piecewise monotone, *i.e.* there are some points χ_j $(j = 0, 1, \cdots, L)$ such that $0 = \chi_0 < \chi_1 < \cdots < \chi_L = R$, and

$$I_j = (\chi_{j-1}, \chi_j), \quad n|_{I_j} \in C^2(\bar{I}_j) \text{ and } n|_{I_j} \text{ is monotone}, \quad j = 1, \cdots, L.$$

First, we take a Fourier transform with respect to y, *i.e.* for $v(x, y) \in L^2(\Omega)$,

$$\hat{v}(x, \xi) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(x, y) e^{-\mathrm{i}\xi y} dy.$$