Journal of Computational Mathematics
Vol.30, No.3, 2012, 223–248.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1109-m3401

LOCAL MULTILEVEL METHODS FOR SECOND-ORDER ELLIPTIC PROBLEMS WITH HIGHLY DISCONTINUOUS COEFFICIENTS*

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Abstract

In this paper, local multiplicative and additive multilevel methods on adaptively refined meshes are considered for second-order elliptic problems with highly discontinuous coefficients. For the multilevel-preconditioned system, we study the distribution of its spectrum by using the abstract Schwarz theory. It is proved that, except for a few small eigenvalues, the spectrum of the preconditioned system is bounded quasi-uniformly with respect to the jumps of the coefficient and the mesh sizes. The convergence rate of multilevel-preconditioned conjugate gradient methods is shown to be quasi-optimal regarding the jumps and the meshes. Numerical experiments are presented to illustrate the theoretical findings.

Mathematics subject classification: 65F10, 65N30.
Key words: Local multilevel method, Adaptive finite element method, Preconditioned conjugate gradient method, Discontinuous coefficients.

1. Introduction

During the last two decades, adaptive finite element methods (AFEM) have been developed very rapidly and have become a popular and powerful tool in numerical solution of partial differential equations (PDEs). Quasi-optimal approximation results can be achieved by mesh adaptivity based on a posteriori error estimates (see, e.g., [6,16,32,36]). In this paper, we also pursue asymptotically optimal methods for computing the solution of the discrete problem. By "optimal" we mean that the computation of the solution asymptotically only requires $O(N)$ operations where N is the number of degrees of freedom (DOFs) on the underlying mesh. Multigrid or multilevel methods are among the most efficient and widely used methods for computing the approximate solution.

The uniform convergence of multigrid methods for conforming finite elements has been widely studied by many authors. We refer to [7–10,12,25,33,43] for a multigrid convergence theory on uniformly refined meshes. Since in AFEM the number of DOFs may not grow exponentially with the mesh levels, as Mitchell pointed out in [31], traditional multigrid methods,

* Received April 28, 2010 / Revised version received July 10, 2011 / Accepted September 1, 2011 / Published online May 7, 2012 /

which perform relaxations on all nodes, may require $O(N^2)$ operations for certain meshes. In order to overcome this issue, local multigrid methods adopt the idea of local smoothing, which restricts relaxations to new elements of each level. Local smoothing turns out to be very efficient on adaptively refined meshes (see, e.g., [26, 46, 48, 50] for elliptic problems with smooth coefficients). Motivated by the recent work of Xu and Zhu [49], we study local multiplicative and additive multilevel algorithms (LMMA and LMAA) for second-order elliptic problems with highly discontinuous coefficients. Different from the works of Chen, Holst, Xu and Zhu [18] for second-order elliptic problems with discontinuous coefficients and Hiptmair and Zheng [27] for Maxwell equations, our algorithm does not reconstruct a virtual refinement hierarchy of meshes. We assume that the meshes are generated by using AFEM based on a posteriori error estimates.

Given a bounded, polygonal or polyhedral domain $\Omega \subset R^d\,(d = 2, 3)$, we consider the following second-order elliptic problem:

$$-\mathrm{div}(\rho(\boldsymbol{x})\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where the source function $f \in L^2(\Omega)$. The coefficient ρ is positive and piecewise constant and may have large jumps in Ω. The homogeneous boundary condition in (1.2) is not essential to our theory and can be replaced with more general boundary conditions. Although problem (1.1)–(1.2) seems to be simple, it plays an important role in many practical applications: such as steady state heat conduction in composite materials, electromagnetism, and multiphase flow.

It is well known that the solution of problem (1.1)–(1.2) may have singularities near reentrant corners of the domain and jumps of the coefficient. The AFEM based on a posteriori error estimates is very efficient to capture local singularities of the solution. A considerable amount of work has been devoted to a posteriori error estimates for such problems. We refer to Bernardi and Verfürth [5], Petzoldt [35], and Chen and Dai [20] for residual-based error estimates, to Luce and Wohlmuth [29] for equilibrated error estimates, and to Cai and Zhang [14] for recovery-based error estimates. For adaptive nonconforming or mixed finite element methods, a posteriori error estimates have been studied by Ainsworth [1, 2] for equilibrated error estimates, by Chen, Xu, and Hoppe [19] for residual-based error estimates, and by Cai and Zhang [15] for recovery-based error estimates.

The purpose of this paper is to study local multilevel solvers for the adaptive finite element discretization of (1.1)–(1.2) and to prove the quasi-optimality of these solvers. It is known that the condition number of the discrete system of the problem (1.1)–(1.2) depends on the jumps of ρ and on the mesh sizes. To reduce the condition number, multigrid methods and domain decomposition methods have been studied for quasi-uniform meshes (see, e.g., [17, 24, 30, 37, 40, 44]). In general, the convergence rate of local multilevel methods depends on the jump of the coefficient, the mesh sizes, or the mesh levels due to the lack of uniform stability estimates for the weighted L^2-projection (see, e.g., [11, 34, 42]). The convergence rate can be improved for some specific scenarios (see, e.g., [22, 23, 34, 45]). Recently, Xu and Zhu (see, e.g., [49, 51]) have proved quasi-uniform convergence of conjugate gradient methods preconditioned by multilevel methods and overlapping domain decomposition methods, respectively.

The objective of this paper is to extend the results of [49] to adaptively refined meshes which are generated by the "newest vertex bisection algorithm" [31, 46]. Using the abstract Schwarz theory, we prove that except for a few small eigenvalues, the *effective condition numbers*, i.e., the ratio of the maximum to the minimum of the remaining eigenvalues of the multilevel-