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AN ANISOTROPIC LOCKING-FREE NONCONFORMING TRIANGULAR FINITE ELEMENT METHOD FOR PLANAR LINEAR ELASTICITY PROBLEM*

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Abstract

The main aim of this paper is to study the nonconforming linear triangular Crouzeix-Raviart type finite element approximation of planar linear elasticity problem with the pure displacement boundary value on anisotropic general triangular meshes satisfying the maximal angle condition and coordinate system condition. The optimal order error estimates of energy norm and L^2-norm are obtained, which are independent of lamé parameter λ. Numerical results are given to demonstrate the validity of our theoretical analysis.

Mathematics subject classification: 65N30, 65N15.
Key words: Planar elasticity, Nonconforming element, Locking-free, Anisotropic meshes.

1. Introduction

We consider the planar linear elasticity problem with the pure displacement boundary value

$$\begin{cases} -\mu\Delta u-(\mu+\lambda)\mathrm{grad}(\mathrm{div}u)=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ,μ are Lamé constants, $\lambda\in(0,+\infty)$, $\mu\in[\mu_1,\mu_2]$, $0<\mu_1<\mu_2$. An equivalent variational formulation to problem (1.1) is

$$\begin{cases} \text{find } u\in V \quad \text{such that}\\ a(u,v)=(f,v) \quad \forall v\in V, \end{cases} \tag{1.2}$$

where $V\subset(H_0^1(\Omega))^2$, $u=(u_1,u_2)$, $f=(f_1,f_2)\in(L^2(\Omega))^2$,

$$a(u,v)=\int_\Omega\{\mu\bigtriangledown u\cdot\bigtriangledown v+(\mu+\lambda)(\mathrm{div}u)(\mathrm{div}v)\}dxdy,\qquad (f,v)=\int_\Omega f\cdot vdxdy.$$

It is well-known that if problem (1.1) is approximated by using standard conforming finite elements as the material becomes nearly incompressible, the numerical solutions converge slowly. Such phenomena have been known as numerical locking. The reason for this lies in that the coefficient of the finite element error estimates is dependent on λ, which will extend to ∞ if $\lambda\to\infty$. More detailed explanation of numerical locking can be found in [1–3].

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In order to overcome the locking phenomena, the special finite element methods were used. One direct approach is to use the mixed formulation, which can be found in [4–7]. The other method is to use the nonconforming finite elements approximation of the pure displacement problem. Based on standard finite element methods, [1] and [2] proved that the linear triangular Crouzeix-Raviart nonconforming element is locking-free. [2] and [8] used the so-called reduced integration methods to take account of a class of triangular and quadrilateral elements. [9] also provided a new method to construct locking-free element, and gave a useful nonconforming incomplete biquadratic rectangular element. However, all the above studies rely on the regularity assumption $h_K/\rho_K \leq C$ or quasi-uniform assumption $h/\tilde{h} \leq C$ [10] of the meshes, where $h_K,\ \rho_K$ denote the diameter and the radius of inscribed circle of the element K respectively, $h = \max_K h_K$, $\tilde{h} = \min_K h_K$, C is a positive constant independent of h. However, in some cases, the solutions of some elliptic problems may have anisotropic behavior in some parts of the solution domain. An obvious idea to reflect this anisotropy is to employ anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. The above assumptions are no longer valid in the case of anisotropic meshes, because the anisotropic elements K are characterized by $h_K/\rho_K \to \infty$, when the limit is considered as $h \to 0$. For the anisotropic elements, the well-known Bramble-Hilbert lemma can not be used directly in estimating the interpolation error. At the same time, the consistency error estimate, the key of the nonconforming finite element analysis, will become very difficult to be dealt with. In recent years, many works have been done to analyze the properties of anisotropic finite elements, especially for the nonconforming finite elements [11–23]. Though [14–18] used the rectangular nonconforming elements to solve the different problems on anisotropic meshes and the Quasi-Wilson element for narrow quadrilateral meshes was discussed in [13], it is difficult to apply these elements to problem (1.1) directly. On the other hand, [12] only discussed the convergence properties for second-order elliptic problem with the nonconforming linear triangular Crouzeix-Raviart type element on anisotropic three-directional meshes. How to extend this element to anisotropic general triangular meshes is still an open problem.

In this paper, we will use the nonconforming linear triangular Crouzeix-Raviart type finite element to approximate problem (1.1) for anisotropic general triangular meshes satisfying the maximal angle condition and coordinate system condition [11]. The optimal order error estimates of energy norm and L^2-norm are obtained by introducing a auxiliary finite element space similar to [12], which are independent of lamé parameter λ. But the analysis is more difficult, and needs more techniques than [12].

The organization of the paper is as follows. In Section 2, we introduce some preliminaries and lemmas. The optimal energy norm and L^2-norm are obtained in Section 3. At last, a numerical example is given to confirm our theoretical analysis in Section 4.

2. Construction of the Nonconforming Anisotropic Element

For the sake of simplicity, we assume that $\Omega \subset R^2$ is a convex polygon composed by a family of triangular meshes J_h, $\Omega = \bigcup_{K \in J_h} \bar{K}$, J_h satisfies the following conditions **(a)** and **(b)** (see Fig. 2.1.), but does not need to satisfy the regularity assumption or quasi-uniform assumption.

(a) Maximal angle condition: There is a constant $\gamma^* < \pi$ (independent of h and $K \in J_h$) such that the maximal interior angle γ of any element K is bounded by γ^*, $\gamma \leq \gamma^*$.