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LINEAR CONVERGENCE OF THE LZI ALGORITHM FOR WEAKLY POSITIVE TENSORS*

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Abstract

We define weakly positive tensors and study the relations among essentially positive tensors, weakly positive tensors, and primitive tensors. In particular, an explicit linear convergence rate of the Liu-Zhou-Ibrahim(LZI) algorithm for finding the largest eigenvalue of an irreducible nonnegative tensor, is established for weakly positive tensors. Numerical results are given to demonstrate linear convergence of the LZI algorithm for weakly positive tensors.

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Key words: Irreducible nonnegative tensor, Weakly positive tensor, Largest eigenvalue, Linear convergence.

1. Introduction

Consider an m-order n-dimensional square tensor $\mathcal{A}$ consisting of n^m entries in the real field R:

$$\mathcal{A} = (A_{i_1\cdots i_m}), \quad A_{i_1\cdots i_m} \in R, \quad 1 \le i_1, \cdots, i_m \le n.$$

Tensors play an important role in physics, engineering, and mathematics. There are many application domains of tensors such as data analysis and mining, information science, image processing, and computational biology [16].

In 2005, Qi [12] introduced the notion of eigenvalues of higher-order tensors, and studied the existence of both complex and real eigenvalues and eigenvectors. Independently, in the same year, Lim [7] also defined eigenvalues and eigenvectors but restricted them to be real. Unlike matrices, eigenvalue problems for tensors are nonlinear. Nevertheless, eigenvalue problems of higher-order tensors have become an important part of a new applied mathematics branch, numerical multilinear algebra, and found a wide range of practical applications, for more references, see [7, 13–15]. The following definition was first introduced by Qi [12] when m is even and $\mathcal{A}$ is symmetric. Chang, Pearson, and Zhang [3] extended it to general square tensors.

Definition 1.1. *Let C be the complex field. A pair $(\lambda, x) \in C\times(C^n\backslash\{0\})$ is called an eigenvalue-eigenvector pair of $\mathcal{A}$, if they satisfy:*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

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where n-dimensional column vectors $\mathcal{A}x^{m-1}$ *and* $x^{[m-1]}$ *are defined as*

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\le i\le n} \qquad \text{and} \qquad x^{[m-1]} := \left(x_i^{m-1}\right)_{1\le i\le n},$$

respectively.

Recently, the largest eigenvalue problem for nonnegative tensors attracted much attention. Chang, Pearson, and Zhang [2] generalized the Perron-Frobenius theorem from nonnegative matrices to irreducible nonnegative tensors. It has numerous applications include multilinear pagerank [7], spectral hypergraph theory [1], and higher-order Markov chains [10]. Pearson [11] introduced the notion of essentially positive tensors and proved that the unique positive eigenvalue is real geometrically simple when the tensor is essentially positive with even order. Here, "real geometrically simple" means that the corresponding real eigenvector is unique up to a scaling constant. Ng, Qi, and Zhou [10] proposed an iterative method for finding the largest eigenvalue of an irreducible nonnegative tensor. The NQZ method in [10] is efficient but it is not always convergent for irreducible nonnegative tensors. Chang, Pearson and Zhang [4] introduced primitive tensors. An essentially positive tensor is a primitive tensor, and a primitive tensor is an irreducible nonnegative tensor, but not vice versa. They established convergence of the NQZ method for primitive tensors. Liu, Zhou, and Ibrahim [9] modified the NQZ method such that the modified algorithm is always convergent for finding the largest eigenvalue of an irreducible nonnegative tensor. Yang and Yang [17] generalized the weak Perron-Frobenius theorem to general nonnegative tensors. Friedland, Gaubert and Han [5] pointed out that the Perron-Frobenius theorem for nonnegative tensors has a very close link with the Perron-Frobenius theorem for homogeneous monotone maps. They introduced weakly irreducible nonnegative tensors and established the Perron-Frobenius theorem for such tensors.

The main contributions of this paper are to introduce the notion of weakly positive tensors, to give the relations among essentially positive tensors, weakly positive tensors, and primitive tensors, and to establish an explicit linear convergence rate of the LZI algorithm in [9] for weakly positive tensors. The linear convergence result is significant for the theory of nonnegative tensors as algorithms for general tensors cannot be so efficient [6, 8].

In Section 3, we introduce weakly positive tensors, and give the relations among essentially positive tensors, weakly positive tensors, and primitive tensors. These tensors are all irreducible nonnegative tensors. An essentially positive tensor is both a primitive tensor and a weakly positive tensor, but not vice versa. A primitive tensor may not be a weakly positive tensor. A weakly positive tensor may also not be a primitive tensor. We give a figure to describe their relationships.

We then establish an explicit linear convergence rate of the LZI algorithm for weakly positive tensors in Section 4. We also show that the LZI algorithm terminates after at most K iterations to produce an ε-approximation of the largest eigenvalue of a weakly positive tensor, where K is a constant related to the fixed accurate tolerance ε. Numerical results are given in Section 5 to demonstrate linear convergence of the LZI algorithm for weakly positive tensors.

2. Preliminaries

Let us first recall some definitions on tensors. An m-order n-dimensional tensor $\mathcal{A}$ is called nonnegative if $A_{i_1\cdots i_m} \ge 0$. We call an m-order n-dimensional tensor the unit tensor, denoted