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A NUMERICAL BOUNDARY EIGENVALUE PROBLEM FOR ELASTIC CRACKS IN FREE AND HALF SPACE*

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Abstract

We present in this paper a numerical method for hypersingular boundary integral equations. This method was developed for planar crack problems: additional edge singularities are known to develop in that case. This paper includes a rigorous error analysis proving the convergence of our numerical scheme. Three types of examples are covered: the Laplace equation in free space, the linear elasticity equation in free space, and in half space.

Mathematics subject classification: 45L05, 65R20, 86-08.
Key words: Hypersingular boundary integral equations, Numerical error analysis, Eigenvalue problems, Faults in free space and half space, Somigliana tensor of the second kind in free space and in half space.

1. Introduction

1.1. The three types of boundary eigenvalue problems studied in this paper

In this paper we study three types of numerical eigenvalue problems using hypersingular boundary integral equations on cracks, either in three dimensional space, or in the lower half space with traction free boundary conditions on the top plane $x_3 = 0$. The first type of problem involves the scalar Laplace operator in free space, cut by a planar fault Γ. Due to the possibility of changing coordinates by rotation and translation we will assume that Γ is contained in the plane $x_3 = 0$. Denoting e_1, e_2, e_3, the natural basis for $\mathbb{R}^3$, we choose the normal vector for Γ to be $n = -e_3$.

We seek to evaluate eigenfunctions f defined in some adequate functional space, ensuring decay at infinity, and satisfying

$$\Delta f = 0, \qquad \text{in } \mathbb{R}^3 \setminus \Gamma, \tag{1.1a}$$

$$[\partial_n f] = 0, \qquad \text{across } \Gamma, \tag{1.1b}$$

$$\partial_n f = \gamma[f]. \tag{1.1c}$$

Here γ is the eigenvalue and brackets indicate jumps across Γ, namely

$$[\partial_n f] = \lim_{(x_1,x_2)\in\Gamma,\, x_3\to 0^+} \partial_n f(x_1,x_2,x_3) - \partial_n f(x_1,x_2,-x_3), \quad (x_1,x_2,0)\in\Gamma.$$

A two dimensional analog to this eigenvalue problem was shown to be relevant to the study of the destabilization of strike slip faults: see [3] and [7]. We are not aware of a straightforward physical interpretation of problem (1.1) in 3D. In the present study this problem is a convenient

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intermediate step between the strike slip case and the fully three dimensional elasticity case. In particular this step is instrumental in comparing our numerical method to others and in deriving an error analysis which is also expected to hold in the subsequent two cases.

The second type of eigenvalue problem considered in this paper involves linear elasticity in free space. The unknown eigenfunctions to be found are vector fields defined in some adequate functional space, ensuring some decay at infinity, and satisfying

$$\mu\Delta\varphi + (\lambda+\mu)\nabla\mathrm{div}\,\varphi = 0, \qquad \text{in } \mathbb{R}^3 \setminus \Gamma, \tag{1.2a}$$

$$[\varphi \cdot e_3] = 0,\ [\varphi \cdot e_2] = 0,\ [T_n\varphi] = 0, \qquad \text{across } \Gamma, \tag{1.2b}$$

$$T_n\varphi \cdot e_1 = \beta[\varphi] \cdot e_1, \tag{1.2c}$$

where the assumptions on Γ are the same as previously, β is the eigenvalue, and $T_n\varphi$ is the usual notation for the stress vector, that is,

$$\sum_{j=1}^{3}(\lambda\,\mathrm{div}\,\varphi\,\delta_{ij} + \mu(\partial_i\varphi_j + \partial_j\varphi_i))n_j.$$

Eqs. (1.2) model a fault in free elastic space undergoing destabilization during which slip (that is, displacement discontinuities) is allowed only in the e_1 direction. A thorough study of this eigenvalue problem, including a formal proof for simplicity of the first eigenspace, was undertaken in [19]. The numerical results shown in that paper are, however, limited by the fact that they were produced by a finite element software package. Questions of numerical convergence, error analysis, and computational domain truncation, were all left unanswered: we propose to address them in this present paper.

The third type of eigenvalue problem considered in this paper involves linear elasticity in half space. Denote $\mathbb{R}^{3-}$ the open set of points (x_1, x_2, x_3) in $\mathbb{R}^3$ such that $x_3 < 0$. We assume that the fault Γ is strictly included in $\mathbb{R}^{3-}$. Traction free conditions are imposed on the top boundary $x_3 = 0$. These conditions are relevant to applications in geophysics. By rotation we can assume that the plane containing Γ has normal direction $n = (n_1, 0, n_3)$. Let t_1 and t_2 be two vectors such that (n, t_1, t_2) forms an orthonormal basis for $\mathbb{R}^3$. The unknown eigenfunctions to be found are vector fields defined in some adequate functional space, ensuring some decay at infinity, and satisfying

$$\mu\Delta\psi + (\lambda+\mu)\nabla\mathrm{div}\,\psi = 0, \qquad \text{in } \mathbb{R}^{3-} \setminus \Gamma, \tag{1.3a}$$

$$T_n\psi = 0, \qquad \text{on } x_3 = 0 \tag{1.3b}$$

$$[\psi \cdot n] = 0,\ [\psi \cdot t_2] = 0,\ [T_n\psi] = 0, \qquad \text{across } \Gamma, \tag{1.3c}$$

$$T_n\psi \cdot t_1 = \beta[\psi \cdot t_1]. \tag{1.3d}$$

Eqs. (1.3) model a fault in free elastic half space undergoing destabilization during which slip (that is displacement discontinuities) is allowed only in the t_1 direction. No forces are applied on the top boundary $x_3 = 0$.

1.2. Outline of our main results and overview of alternative computational methods found in the literature

The main achievement of this paper is to provide a numerical method for each of the problems (1.1), (1.2), (1.3) which relies on boundary integral formulations. The advantage of