Journal of Computational Mathematics
Vol.29, No.4, 2011, 474–490.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1103-m3261

OPERATOR SPLITTING SCHEMES FOR THE NON-STATIONARY THERMAL CONVECTION PROBLEMS*

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Abstract

In this work, a new numerical scheme is proposed for thermal/isothermal incompressible viscous flows based on operator splitting. Unique solvability and stability analysis are presented. Some numerical result are given, which show that the proposed scheme is highly efficient for the thermal/isothermal incompressible viscous flows.

Mathematics subject classification: 35Q30, 74S05.
Key words: θ scheme, Stability, Isothermal incompressible viscous flows.

1. Introduction

For the time-dependent thermal and isothermal incompressible viscous flow governed by the Boussinesq and the Navier-Stokes equations, the numerical approximation requires the determination of the fluid's velocity, pressure and temperature. A direct approximation technique requires the solution of a very large nonlinear system of equations at each time step. The fractional step θ-method, developed by Glowinski in [1], is an appealing numerical approximation technique [2–4]. It updates the velocity/pressure and temperature using several sub-steps, which leads to decoupling the difficulties associated with the non-linearities and incompressibility condition, thereby reducing the size of the algebraic systems at each sub-step.

In the last decades, a number of numerical methods have been proposed for the numerical simulation of thermal/isothermal incompressible viscous flows. In [5, 6], the numerical simulation is performed in the stream function-vorticity formulation. Hortmann *et al.* [7] considers the same problem, but solves it with finite volumes in primitive variables for the stationary case. Le Qurin [8] provided accurate transient solutions at high Rayleigh number by using pseudo-spectral discretization with Chebyshev polynomials. In [9], numerical schemes for time-dependent incompressible viscous fluid flow, thermally coupled under the Boussinesq approximation, are presented. The schemes combine an operator splitting in the time discretization and linear finite elements in the space discretization.

In this paper, a new numerical scheme is proposed, which combines an θ scheme in time discretization and linear finite elements in the space discretization. The unique solvability and stability analysis of the proposed scheme are presented. Numerical experiments show that the scheme is efficient for simulating of thermal/isothermal incompressible viscous flows.

The remainder of this paper is organized as follows: in the next section, the mathematical model and some basic notation are introduced. In Section 3, we describe the fractional step θ-time stepping scheme which consists of three steps in each interval of time and a detailed description of the numerical solution of the subproblems is present. In Section 4, the unique

* Received November 16, 2009 / Revised version received October 11, 2010 / Accepted December 28, 2010 / Published online June 27, 2011 /

solvability is presented. In Section 5, the proof of stability of the fractional step θ scheme is given. In Section 6, some numerical result are given to illustrate the theoretical results. Some concluding remarks are given in the final section.

2. The Mathematical Model

Under the well-known Boussinesq approximation, the time-dependent flow is governed by the non-dimensional equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \triangledown)\mathbf{u} + \triangledown p = \lambda \mathbf{g} T, \\ \triangledown \cdot \mathbf{u} = 0, \\ \dfrac{\partial T}{\partial t} - \xi \Delta T + (\mathbf{u} \cdot \triangledown) T = 0, \end{cases} \tag{2.1}$$

where $\mathbf{x} \in \Omega \subset R^n$ (n=2, 3), Ω is a bounded region in R^n with a sufficiently regular boundary $\partial\Omega$. The unknowns are the vector function $\mathbf{u}$ (velocity), the scalar function p (pressure) and the scalar function T (temperature). The dimensionless parameters Re, Ra, Pr are the Reynolds, Rayleigh and Prandtl number, respectively. g is the gravity vector $\mathbf{g} = (0, 1)$, $\nu = 1/Re$ is the viscosity, and we also define $\lambda = (Ra)/(PrRe^2)$, $\xi = 1/(RePr)$.

For the sake of completeness, Eqs. (2.1) should be supplemented with appropriate initial and boundary condition:

$$\begin{cases} \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega\, (\nabla \cdot \mathbf{u}_0 = 0), \\ T(\mathbf{x}, 0) = T_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u} = 0, & \text{on}\, \partial\Omega, \\ T = T_0, & \text{on}\, \partial\Omega, \end{cases} \tag{2.2}$$

Remark 2.1. It follows from [10] that

(1) The coupling between the first and the third equation in (2.1) involving Re corresponding to mixed convection. For natural convection, $Re = 1$ is taken.

(2) For the Navier-Stokes equations, there is no coupling with the thermal energy equation, and the right hand side of the first formula in (2.1) involves a concentration of external forces $\mathbf{f}$ independent of T. Consequently, it is independent of parameters Ra, Pr and Re.

Next, we will introduce some notations and results which will be frequently used in this paper. Let $(\cdot, \cdot)$, $\|\cdot\|$ denote, the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^n$, respectively. The spaces $H_0^1(\Omega)$ and $H_0^1(\Omega)^n$ are equipped with their usual norm:

$$\|\mathbf{u}\|_1^2 = \int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}.$$

The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. We also use $\langle \cdot, \cdot \rangle$ to denote the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ for all $s > 0$.