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VARIATIONAL DISCRETIZATION OF PARABOLIC CONTROL PROBLEMS IN THE PRESENCE OF POINTWISE STATE CONSTRAINTS*

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Abstract

We consider a parabolic optimal control problem with pointwise state constraints. The optimization problem is approximated by a discrete control problem based on a discretization of the state equation by linear finite elements in space and a discontinuous Galerkin scheme in time. Error bounds for control and state are obtained both in two and three space dimensions. These bounds follow from uniform estimates for the discretization error of the state under natural regularity requirements.

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1. Introduction

Optimal control of time-dependent production processes plays an important role in many practical applications such as crystal growth [10,16,17] and cooling of glass melts [5,22]. These processes are frequently described by systems of partial differential equations involving the temperature as a system variable. A need to avoid overheating of the device or to prevent solidification/melting at the wrong places then naturally leads to pointwise bounds on the temperature variable. The introduction of pointwise state conditions however yields adjoint variables and multipliers which only admit low regularity complicating the analysis of the necessary first order conditions. These problems need to be taken into account in the numerical approximation and necessitate the development of tailored discrete concepts.

In the present work we consider an optimal control problem for the heat equation and with pointwise bounds on the state. The optimization problem is approximated using variational discretization [14] combined with linear finite elements in space and a discontinuous Galerkin scheme in time for the discretization of the state equation, compare [15, Chapter 3]. Our main result are L^2–error estimates for the optimal state and the optimal control. To derive these bounds, uniform estimates for the discretization error of the state under natural regularity requirements are proved. For the numerical analysis of the optimal control problem we use an approach which avoids error estimates for the adjoint state and which was developed in [7],

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[15, Chapter 3] for the analysis of elliptic optimal control problems with state and gradient constraints.

Although a lot of contributions are known on elliptic optimal control problems with pointwise bounds on the state, see e.g. [4,6,7,15,18,19,23], to the best of the authors knowledge numerical analysis of parabolic optimal control problems with pointwise bounds in space-time for the state has not yet been considered in the literature. In this work we present the numerical analysis for our result of Theorem 4.1 which we already announced in [11]. However, there are some contributions on the analysis of related control problems. In [20] Lavrentiev regularization of state constrained parabolic control problems is investigated, optimal control problems with pointwise state constraints in time and averaged state constraints in space are considered in [1]. We note that numerical analysis for this particular setting is announced by Vexler in [11]. Optimality conditions for parabolic optimal control problems in the presence of state constraints are investigated in [8], where further references on analysis aspects of state constrained parabolic control problems can be found.

2. The Optimal Control Problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded convex polygonal domain, $T > 0$, $\Omega_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$. Let us consider the initial boundary value problem

$$y_t - \Delta y = f, \qquad \text{in } \Omega_T, \tag{2.1}$$

$$\frac{\partial y}{\partial \nu} = 0, \qquad \text{on } \Gamma_T, \tag{2.2}$$

$$y(\cdot, 0) = y_0, \qquad \text{in } \Omega. \tag{2.3}$$

It is well–known that for given $f \in L^2(0, T; L^2(\Omega)), y_0 \in H^1(\Omega)$ problem (2.1)–(2.3) has a unique solution $y \in C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$. In what follows we shall keep the initial datum y_0 fixed and denote by $\hat{y}$ the solution of (2.1)–(2.3) corresponding to $f \equiv 0$. This allows us to write the solution of (2.1)–(2.3) in the form

$$y = \mathcal{G}(f) = \hat{y} + \mathcal{G}_0(f), \tag{2.4}$$

where $\mathcal{G}_0(f)$ is the linear operator that assigns to f the solution of (2.1)–(2.3) for $y_0 \equiv 0$. Note that if $f \in L^2(0, T; H^1(\Omega))$ and

$$y_0 \in H^2(\Omega), \qquad \text{with } \frac{\partial y_0}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \tag{2.5}$$

then we have

$$y \in W := \Big\{ w \in C^0([0, T]; H^2(\Omega)) \mid w_t \in L^2(0, T; H^1(\Omega)) \Big\},$$

and

$$\max_{0 \le t \le T} \|y(t)\|_{H^2}^2 + \int_0^T \|y_t(t)\|_{H^1}^2 dt \le C\Big(\|y_0\|_{H^2}^2 + \int_0^T \|f(t)\|_{H^1}^2 dt \Big). \tag{2.6}$$

We remark that $W \subset C^0(\overline{\Omega_T})$ since we have the continuous embedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $d = 2, 3$.

Next, suppose that the functions $f_1, \cdots, f_m \in H^1(\Omega) \cap L^\infty(\Omega)$ are given and define $U := L^2(0, T; \mathbb{R}^m)$ as well as $B : U \to L^2(0, T; H^1(\Omega))$ by

$$(Bu)(x, t) := \sum_{i=1}^m u_i(t) f_i(x), \quad (x, t) \in \Omega_T. \tag{2.7}$$