Journal of Computational Mathematics
Vol.28, No.6, 2010, 826–836.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1004-m2775

LOCALLY STABILIZED FINITE ELEMENT METHOD FOR STOKES PROBLEM WITH NONLINEAR SLIP BOUNDARY CONDITIONS*

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Abstract

Based on the low-order conforming finite element subspace (V_h, M_h) such as the P_1-P_0 triangle element or the Q_1-P_0 quadrilateral element, the locally stabilized finite element method for the Stokes problem with nonlinear slip boundary conditions is investigated in this paper. For this class of nonlinear slip boundary conditions including the subdifferential property, the weak variational formulation associated with the Stokes problem is an variational inequality. Since (V_h, M_h) does not satisfy the discrete inf-sup conditions, a macroelement condition is introduced for constructing the locally stabilized formulation such that the stability of (V_h, M_h) is established. Under these conditions, we obtain the H^1 and L^2 error estimates for the numerical solutions.

Mathematics subject classification: 35Q30.
Key words: Stokes Problem, Nonlinear Slip Boundary, Variational Inequality, Local Stabilized Finite Element Method, Error Estimate.

1. Introduction

Numerical simulation for the incompressible flow is the fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that the mathematical model of viscous incompressible fluid with homogeneous boundary conditions is the Navier-Stokes equations which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T, \\ \text{div} u = 0 & \text{in } Q_T \\ u(0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0, T], \end{cases}$$

where $Q_T = (0, T] \times \Omega$, $0 < T \leq +\infty$, $\Omega \subset \mathbb{R}^n, n = 2, 3$, is a bounded convex domain; $u(t, x)$ and $f(t, x)$ are vector functions representing the flow velocity and the external force, respectively; $p(t, x)$ is a scalar function representing the pressure. The viscous coefficient $\mu > 0$ is a positive constant. The solenoidal condition means that the fluid is incompressible.

* Received March 10, 2008 / Revised version received September 29, 2009 / Accepted October 20, 2009 / Published online August 9, 2010 /

Note that the velocity u and the pressure p are coupled by the solenoidal condition $\mathrm{div}u = 0$ which makes that it is difficult to solve the Navier-Stokes equations. Some popular techniques to overcome this difficulty are to relax the solenoidal condition in an appropriate way which leads to a pesudo-compressible system, such as the penalty method, the artificial compressible method, the pressure stabilized method and the projection method, see, e.g., [1,2,10-14,18-21].

In this paper, we will consider Stokes problem

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega, \\ \mathrm{div}u = 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

with the nonlinear slip boundary conditions

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex domain; $\Gamma \cap S = \emptyset, \overline{\Gamma \cup S} = \partial\Omega$; g is a scalar function; $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, with n the unit vector of the external normal to S; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential components of the stress vector σ defined by

$$\sigma_i = \sigma_i(u,p) = (\mu e_{ij}(u) - p\delta_{ij})n_j.$$

Here

$$e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}, i, j = 1, 2.$$

The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a, whose definition will be given in next section.

The boundary conditions (1.2) are introduced by Fujita in [4], who investigated some hydrodynamics problems under nonlinear boundary conditions, such as leak and slip boundary involving subdifferential property. These types of boundary conditions appear in the modeling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks. Fujita in [5] showed the existence and uniqueness of weak solution to the Stokes problem with slip boundary conditions (1.2). Subsequently, Saito in [17] showed the regularity of the weak solution by using Yosida's regularized method and finite difference quotients method. Other theoretical results about the Stokes problems with nonlinear subdifferential boundary conditions can be found in [6-8,16]. We remark that the steady homogeneous and inhomogeneous Stokes system with linear slip boundary conditions without subdifferential property have recently been studied from the theoretical view point by Veiga in [22-24].

The aim of this paper is to extend the locally pressure stabilized finite element method, which is introduced by Kechkar & Silvester in [14] and developed by He *et al.* for the Navier-Stokes equations in [10-13], and to the problem (1.1)-(1.2). This method bases on the lower order conforming finite element subspace (V_h, M_h) such as P_1-P_0 triangle element (linear velocity, constant pressure) or the Q_1-P_0 quadrilateral element (bilinear velocity, constant pressure). Since (V_h, M_h) does not satisfy the discrete inf-sup conditions, a macroelement condition is introduced for constructing the locally stabilized formulation such that the stability of (V_h, M_h) is established. Under these conditions, we show that if the true solution $(u,p) \in H^2(\Omega)^2 \cap V \times H^1(\Omega) \cap M$, then the following H^1 and L^2 error estimates hold:

$$\|u - u_h\|_V + \|p - p_h\| \le ch^{\frac{1}{2}}, \tag{1.3}$$

$$\|u - u_h\| \le ch^{\frac{3}{2}}, \tag{1.4}$$