Journal of Computational Mathematics
Vol.28, No.5, 2010, 711–724.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m2809

VARIABLE MESH FINITE DIFFERENCE METHOD FOR SELF-ADJOINT SINGULARLY PERTURBED TWO-POINT BOUNDARY VALUE PROBLEMS *

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Abstract

A numerical method based on finite difference method with variable mesh is given for self-adjoint singularly perturbed two-point boundary value problems. To obtain parameter-uniform convergence, a variable mesh is constructed, which is dense in the boundary layer region and coarse in the outer region. The uniform convergence analysis of the method is discussed. The original problem is reduced to its normal form and the reduced problem is solved by finite difference method taking variable mesh. To support the efficiency of the method, several numerical examples have been considered.

Mathematics subject classification: 45Exx, 65L10, 65L12, 65L50, 65L70.
Key words: Singularly perturbed boundary value problems, Finite difference method, Boundary layer, Parameter uniform-convergence, Variable mesh.

1. Introduction

The problem in which a small parameter multiplies to the highest derivative arise in various fields of science and engineering, for instance, fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, hydrodynamics etc. A large number of papers and books have been published describing various methods for solving singular perturbation problems, see, e.g., Axelsson *et al.* [2], Bellman [3], Bender and Orszag [4], Cole and Kevorkian [5], Eckhaus [7], Hamker and Miller [10], O'Malley [13], Nayfeh [16], Van Dyke [22]

Niijima [17] gave uniformly second order accurate difference schemes for reaction-diffusion equations, whereas Miller [14] gave sufficient condition for the uniform first-order convergence of a general three-point difference scheme. Parameter-uniform numerical methods [9, 15] are methods whose numerical approximations U^N satisfy error bounds of the form

$$\|u_\epsilon - U^N\| \le C\vartheta(N), \quad \vartheta(N) \to 0 \ \text{as} \ N \to \infty,$$

where u_ϵ is the solution of the continuous problem, $\|.\|$ is the maximum pointwise norm, N is the number of mesh points (independent of ϵ) used and C is a positive constant which is independent of both ϵ and N. In other words, the numerical approximations U^N converge to u_ϵ for all values of parameter ϵ in the range $0 < \epsilon \ll 1$.

It is well-known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ϵ is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy does not depend on the parameter value ϵ, i.e., the methods are convergent

* Received April 11, 2008 / Revised version received July 30, 2009 / Accepted October 20, 2009 / Published online May 1, 2010 /

ϵ-uniformly [6,8,20]. In this paper, the strategy and the proposed method based on a suitably designed fitted mesh has been shown to converge with $\vartheta(N) = N^{-1} \ln N$.

In this paper, we consider the following self-adjoint singularly perturbed two-point boundary value problem

$$Ly \equiv -\epsilon(p(x)y')' + q(x)y = f(x), \tag{1.1}$$

for $0 \leq x \leq 1$ with the natural boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta, \tag{1.2}$$

where α, β are given constants and ϵ is a small positive parameter $(0 < \epsilon \ll 1)$. Further assume that the coefficients $p(x)$, $q(x)$ and the function $f(x)$ are smooth and satisfy

$$p(x) \geq \eta_1 > 0, \quad p'(x) \geq 0, \quad q(x) \geq \eta_2 > 0.$$

Under these conditions, the operator L admits the maximum principle [18].

In general finding the numerical solution of a second order boundary value problem with y' term is more difficult as compare to a second order boundary value problem without y' term, therefore we first reduce (1.1) to its normal form and then the reduced problem is solved by finite difference scheme using arithmetic mesh.

Briefly, outline is as follows. In Section 2, we give description of the method. The derivation of the difference scheme has been given in Section 3. The idea how to choose the mesh has been given in Section 4, whereas the parameter uniform-convergence of the scheme is given in Section 5. To demonstrate the efficiency of the method some numerical experiments have been solved in Section 6 and finally the conclusion has been presented in Section 7.

2. Description of the Method

Eq. (1.1) can be rewritten as

$$y'' + P(x)y' + Q(x)y = F(x), \tag{2.1}$$

where

$$P(x) = \frac{p'(x)}{p(x)}, \quad Q(x) = -\frac{q(x)}{\epsilon p(x)} \quad \text{and} \quad F(x) = -\frac{f(x)}{\epsilon p(x)}.$$

By the transformation

$$y(x) = U(x)V(x), \tag{2.2}$$

Eq. (2.1) can be written as its normal form:

$$V''(x) + A(x)V(x) = G(x), \tag{2.3}$$

with

$$V(0) = \frac{y(0)}{U(0)} = \gamma, \quad V(1) = \frac{y(1)}{U(1)} = \delta, \quad \gamma, \delta \in \mathbb{R}, \tag{2.4}$$

where

$$A(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2,$$

$$G(x) = F(x) \exp\left(\frac{1}{2}\int^x P(\zeta)\, d\zeta\right), \quad U(x) = \exp\left(-\frac{1}{2}\int^x P(\zeta)\, d\zeta\right).$$