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UNIFIED ANALYSIS OF TIME DOMAIN MIXED FINITE ELEMENT METHODS FOR MAXWELL'S EQUATIONS IN DISPERSIVE MEDIA*

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Abstract

In this paper, we consider the time dependent Maxwell's equations when dispersive media are involved. The Crank-Nicolson mixed finite element methods are developed for three most popular dispersive medium models: the isotropic cold plasma, the one-pole Debye medium and the two-pole Lorentz medium. Optimal error estimates are proved for all three models solved by the Raviart-Thomas-Nédélec spaces. Extensions to multiple pole dispersive media are presented also.

Mathematics subject classification: 65N30, 35L15, 78-08.
Key words: Maxwell's equations, Dispersive media, Mixed finite element method.

1. Introduction

The dispersive medium is characterized by a frequency-dependent susceptibility or permittivity, so that monochromatic waves of different frequencies travel in the medium at different velocities and undergo different attenuations. The most common dispersive media include biological tissue, ionosphere, water, soil, snow, ice, plasma, optical fibers and radar absorbing materials. Hence the study of wave or pulse propagation in dispersive media is important in many applications.

Starting early 1990's, considerable attention has been devoted to numerical modeling of wave propagation in dispersive media. Approaches such as the recursive convolution method and auxiliary differential equation method have been developed under the framework of the finite-difference time-domain (FDTD) method, details and early references can be found in books [19, Ch.8] and [29, Ch.9]. However, due to its complexity, the time-domain finite element method (TDFEM) for the dispersive media has not explored until 2001 by Jiao and Jin [18]. Their TDFEM is based on the second-order vector wave equation. Recently, the time-domain discontinuous Galerkin method has been investigated by Lu *et al.* [24] by solving the first-order Maxwell's equations directly. The one dimensional TDFEM was studied for Debye and Lorentz dispersive media by Bank *et al.* [4] recently.

Since 1980's, there has been a growing interest in finite element analysis of Maxwell's equations (e.g. [3, 5–9, 12–15, 17, 23, 27, 28, 32]). However, almost all studies are restricted to the simple medium case. Very recently, we initiated the error analysis of TDFEM for dispersive

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media [20–22]. In [22], we discussed the superconvergence results for some semi-discrete schemes developed for dispersive medium models. While in [20], we analyzed the backward Euler mixed finite element methods (FEMs) for three most popular dispersive medium models. In [21], we studied the backward Euler scheme for the vector wave equation resulting from the isotropic non-magnetized cold plasma model. In all our previous work, the FEMs are all built on the integro-differential equations. In this paper, we propose some Crank-Nicolson mixed FEMs directly on the governing equations without introducing integral terms. It turns out that this algorithm is simpler and the error analysis can be beautifully carried through by skillful manipulations. Here we provide a unified optimal error analysis for all three popular dispersive medium models.

We conclude the section with an outline of the remainder of the paper. In next section, we consider the single pole Debye medium solved by the Crank-Nicolson mixed method using the lowest Raviart-Thomas-Nédélec (RTN) space. Optimal error estimates are proved under proper regularity assumptions. Then we extend the results to the multiple pole Debye medium. In Section 3, we generalize the numerical scheme and error analysis to both the two-pole and multiple pole Lorentz media. Section 4 is devoted to the isotropic cold plasma model. Similar numerical scheme and results are presented. Finally, we conclude the paper in Section 5.

In this paper, C (sometimes with sub-index) denotes a generic constant, which is independent of the finite element mesh size h and time step size τ. We also use some common notation

$$H^{\alpha}(\text{curl};\Omega)=\left\{\boldsymbol{v}\in(H^{\alpha}(\Omega))^3;\ \ \nabla\times\boldsymbol{v}\in(H^{\alpha}(\Omega))^3\right\},$$
$$H_0(\text{curl};\Omega)=\left\{\boldsymbol{v}\in H(\text{curl};\Omega);\ \ \mathbf{n}\times\boldsymbol{v}=0\ \text{ on }\ \partial\Omega\right\},$$

where $\alpha\geq 0$ is a real number, and Ω is a bounded and convex Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. When $\alpha=0$, we simply denote $H^0(\text{curl};\Omega)=H(\text{curl};\Omega)$. Let $(H^{\alpha}(\Omega))^3$ be the standard Sobolev space equipped with the norm $\|\cdot\|_{\alpha}$ and semi-norm $|\cdot|_{\alpha}$. In particular, $\|\cdot\|_0$ will mean the $(L^2(\Omega))^3$-norm. Also $H(\text{curl};\Omega)$ and $H^{\alpha}(\text{curl};\Omega)$ are equipped with the norm

$$\|\boldsymbol{v}\|_{0,\text{curl}}=\left(\|\boldsymbol{v}\|_0^2+\|\text{curl }\boldsymbol{v}\|_0^2\right)^{1/2},$$
$$\|\boldsymbol{v}\|_{\alpha,\text{curl}}=\left(\|\boldsymbol{v}\|_{\alpha}^2+\|\text{curl }\boldsymbol{v}\|_{\alpha}^2\right)^{1/2}.$$

Finally, we denote $C^m(0,T;X)$ the space of m times continuously differentiable functions from $[0,T]$ into the Hilbert space X.

2. Debye Medium

For the single pole Debye medium model, we have the governing equations [20, 30]

$$\epsilon_0\epsilon_\infty\frac{\partial\boldsymbol{E}}{\partial t}=\nabla\times\boldsymbol{H}-\frac{(\epsilon_s-\epsilon_\infty)\epsilon_0}{t_0}\boldsymbol{E}+\frac{1}{t_0}\boldsymbol{P},\tag{2.1}$$
$$\mu_0\frac{\partial\boldsymbol{H}}{\partial t}=-\nabla\times\boldsymbol{E},\tag{2.2}$$
$$\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0}\frac{\partial\boldsymbol{P}}{\partial t}+\frac{1}{(\epsilon_s-\epsilon_\infty)\epsilon_0 t_0}\boldsymbol{P}=\frac{1}{t_0}\boldsymbol{E},\tag{2.3}$$